Relations between degree of satisfaction and consistency index in fuzzy data AHP

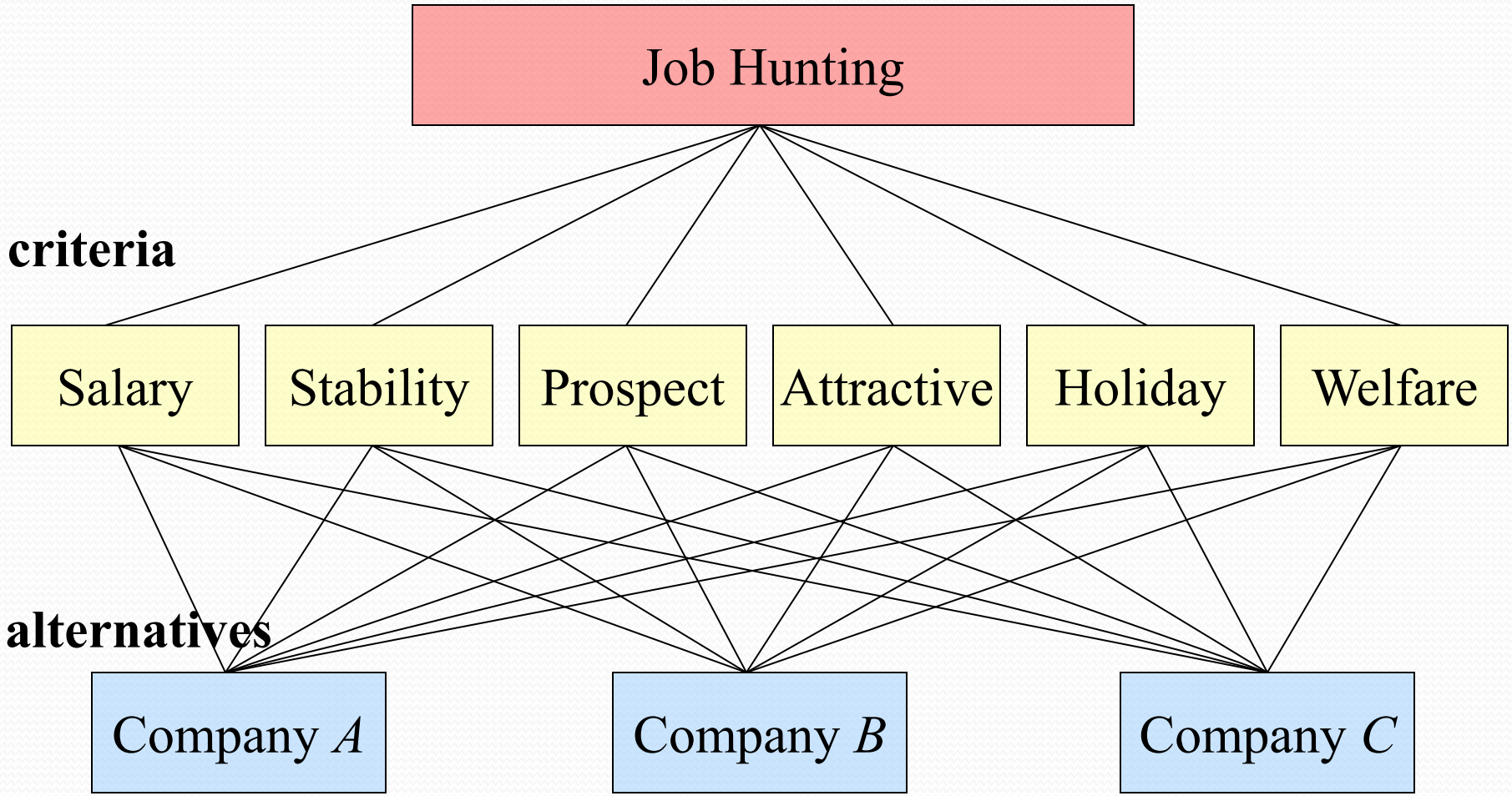
Shin-ichi Ohnishi, Takahiro Yamanoi
Hokkai-Gakuen University, Japan
Hokkaido University, Japan
ohnishi@hgu.jp

Introduction

- Among many extended versions of Analytic Hierarchy Process (AHP), fuzzy data AHP is a one of the most useful method, because it can reflect vagueness of decision maker's answers using fuzzy reciprocal matrix as data.
- However, it is not so easy to keep reliability of data because of worsening of consistency index in AHP.
- In this research, we use a sensitivity analysis to discuss relations between degree of satisfaction and consistency index in fuzzy data AHP through numerical experiments.

Hierarchical structure in AHP

1. **Representation by a hierarchy**
2. Pairwise comparison matrices
3. (Consistency check)
4. Local weights of criteria
5. Global weights of alternative



Example

weights are normalized eigenvector corresponding to maximum eigenvalue

(P2)pairwise comparison matrix

$$
\begin{array}{c}
\\ \text{Salary} \\ \text{Stability} \\ \text{Prospect} \\ \text{Attractive} \\ \text{Holiday} \\ \text{Welfare}
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \text{Salary} & \text{Stability} & \text{Prospect} & \text{Attractive} & \text{Holiday} & \text{Welfare} \end{array} \\
\begin{pmatrix}
1 & 1/5 & 1/5 & 1/5 & 1/2 & 1/3 \\
 & 1 & 3 & 4 & 7 & 5 \\
 & & 1 & 3 & 6 & 5 \\
 & & & 1 & 7 & 3 \\
 & & & & 1 & 1/5 \\
 & & & & & 1
\end{pmatrix}
\end{array}
$$

- reciprocal data matrix
- **checking consistency**

C.I.=0.13

(P4)weights of criteria

$$
\begin{pmatrix} \text{Salary} \\ \text{Stability} \\ \text{Prospect} \\ \text{Attractive} \\ \text{Holiday} \\ \text{Welfare} \end{pmatrix}
=
\begin{pmatrix} 0.04 \\ 0.41 \\ 0.26 \\ 0.16 \\ 0.04 \\ 0.09 \end{pmatrix}
$$

(P4)weights of alternatives with respect to activities

	Company A	Company B	Company C
Salary	0.158	0.766	0.076
Stability	0.121	0.273	0.606
Prospect	0.180	0.778	0.042
Attractive	0.070	0.751	0.178
Holiday	0.157	0.249	0.594
Welfare	0.121	0.115	0.764

(P5)total weights of alternatives

$$
\begin{pmatrix} \text{Company } A \\ \text{Company } B \\ \text{Company } C \end{pmatrix}
=
\begin{pmatrix} 0.10 \\ 0.45 \\ 0.45 \end{pmatrix}
$$

Consistency index of the pairwise comparison matrix *A* (checking reliability of data, C.I.)

$$\text{C.I.} = \frac{\lambda_A - n}{n - 1}$$

where

A : comparison matrix with order n

λ_A : maximum eigenvalue (Frobnius root) of A

C.I. > 0.1, ⇒bad consistency

→ re-evaluate again **using sensitivity analysis**

Sensitivity analysis of eigenvalue for consistency

$$A(\varepsilon) = A + \varepsilon D_A$$

perturbation $D_A = (a_{ij} d_{ij})$

$A = a_{ij} \; (i, j = 1, \ldots, n)$

$\lambda(\varepsilon)$: eigenvalue of $A(\varepsilon)$

$$\boxed{\lambda(\varepsilon) = \lambda_A + \varepsilon \lambda^{(1)} + o(\varepsilon)}$$

eigenvalue of perturbed matrix

eigenvalue of original matrix

fluctuation

$\boldsymbol{w}_1 = (w_{1i})$: eigenvalue of A

$\boldsymbol{w}_2 = (w_{2i})$: eigenvalue of A'

$$\lambda^{(1)} = \frac{1}{\boldsymbol{w}_1{}^{\mathrm{T}} \boldsymbol{w}_2} \sum_i^n \sum_j^n w_{2i} a_{ij} w_{1j} d_{ij}$$

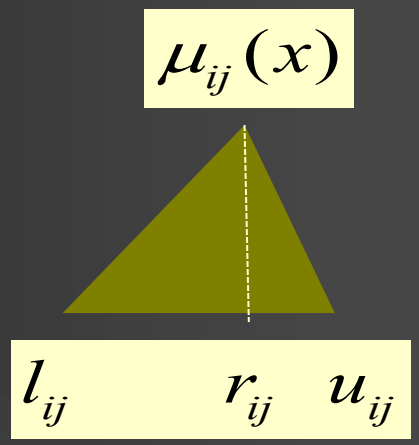
Components of fuzzy data matrix
（Ohnishi, Dubois, Prade 2006）

fuzzy data

$$\tilde{r}_{ij} = (l_{ij}, r_{ij}, u_{ij})_{\Delta}$$

$$\mu_{ij}(r_{ij}) = 1$$

$$\mu_{ij}(l_{ij}) = \mu_{ij}(u_{ij}) = 0$$



reciprocity

$$\mu_{ij}(r) = \mu_{ji}(1/r)$$

$$\mathrm{core}(\tilde{r}_{ji}) = 1/r_{ij}$$

$$\mathrm{supp}(\tilde{r}_{ji}) = [1/u_{ij}, 1/l_{ij}]$$

Optimal degree of satisfaction and weight of fuzzy data AHP

$$\alpha^* \equiv \max_{w_1,\ldots,w_n} \min_{i,j} \left\{ \mu_{ij}\left(\frac{w_i}{w_j} \right) \right\}$$

If all $\tilde{r}_{ij}$ $(i < j)$ are triangular fuzzy numbers $(l_{ij}, r_{ij}, u_{ij})_\Delta$,

[NLP]

Maximize α

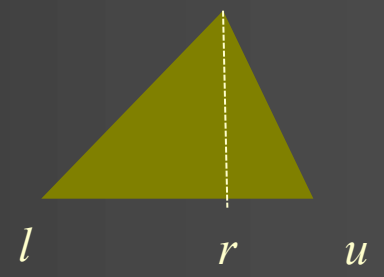
$$w_j \{ l_{ij} + \alpha(r_{ij} - l_{ij}) \} \le w_i \le w_j \{ u_{ij} + \alpha(r_{ij} - u_{ij}) \}$$

$$\sum_i^n w_i = 1 \qquad (i, j = 1, \ldots, n)$$

Choice of crisp value for sensitivity analysis of consistency on fuzzy data

fuzzy data

$$\tilde{r}_{ij} = (l_{ij}, r_{ij}, u_{ij})_{\Delta}$$



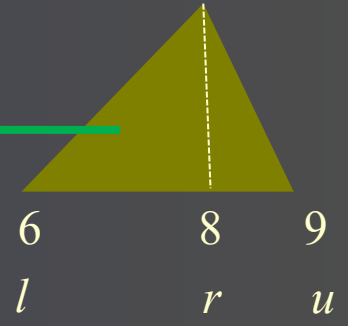
(1) core r of each component of matrix
(2) support set（interval $[l,u]$） of each component
(3) interval $[l,u]$ corresponding to α-cut-set (interval $[L,U]$)

✓ an endpoint of α-cut-set for calculating crisp weight must be meaningful

example

fuzzy reciprocal data matrix

1	$(1, 3, 5)_\triangle$	$(2, 5, 7)_\triangle$	$(6, 8, 9)_\triangle$
	1	$(1, 2, 4)_\triangle$	$(2, 4, 5)_\triangle$
		1	$(0.5, 2, 3)_\triangle$
			1



α-cut interval matrix

1	[2.42,3.58]	[4.13,5.58]	[7.42,8.29]
	1	[1.71,2.58]	[3.42,4.29]
		1	[1.57,2.29]
			1

$[L_{ij},\ U_{ij}]$

$\alpha^*=0.711$

crisp matrix

1	2.42	5.28	8.29
	1	2.18	3.42
		1	1.57
			1

w_1	0.581
w_2	0.240
w_3	0.110
w_4	0.070

Numerical experiment: checking result of sensitivity analysis

crisp matrix for analysis

$$A_N = \begin{pmatrix} 1 & 1 & 5.28 & \boxed{9} \\ & 1 & 2.18 & 2 \\ & & 1 & 0.5 \\ & & & 1 \end{pmatrix}$$

$$\text{C.I.}_N = \frac{\lambda_l - n}{n-1} = 0.106 > 0.1$$

result of sensitivity analysis

-0.050	0.035	**0.057**
	-0.030	-0.020
		-0.037

the biggest absolute value has most influence

$$A_{N2} = \begin{pmatrix} 1 & 1 & 5.28 & \boxed{8} \\ & 1 & 2.18 & 2 \\ & & 1 & 0.5 \\ & & & 1 \end{pmatrix}$$

$$\text{C.I.}_{N2} = \frac{\lambda_l - n}{n-1} = 0.092 < 0.1$$

fuzzy data matrix

1	$(1, 3, 5)_\triangle$	$(2, 5, 7)_\triangle$	$(6, 8, 8)_\triangle$
	1	$(1, 2, 4)_\triangle$	$(2, 4, 5)_\triangle$
		1	$(0.5, 2, 3)_\triangle$
			1

Conclusions and Summary

Sensitivity analysis of consistency for fuzzy data AHP

- **As a choice of crisp value for sensitivity analysis**
  - ✓ Selection of an endpoint of α-cut-set for calculating crisp weight must be more meaningful than using other value.
- **Relations between degree of satisfaction and consistency index in fuzzy data AHP through numerical experiments.**
  - ✓ It has an impact on improving the consistency index C.I. by changing an endpoint of the support set of fuzzy value.
  - ✓ However, whether it leads to an improvement of the degree of satisfaction α is another matter.
  - ✓ All changes direction of endpoints cause to reduce the width of the support set in our example