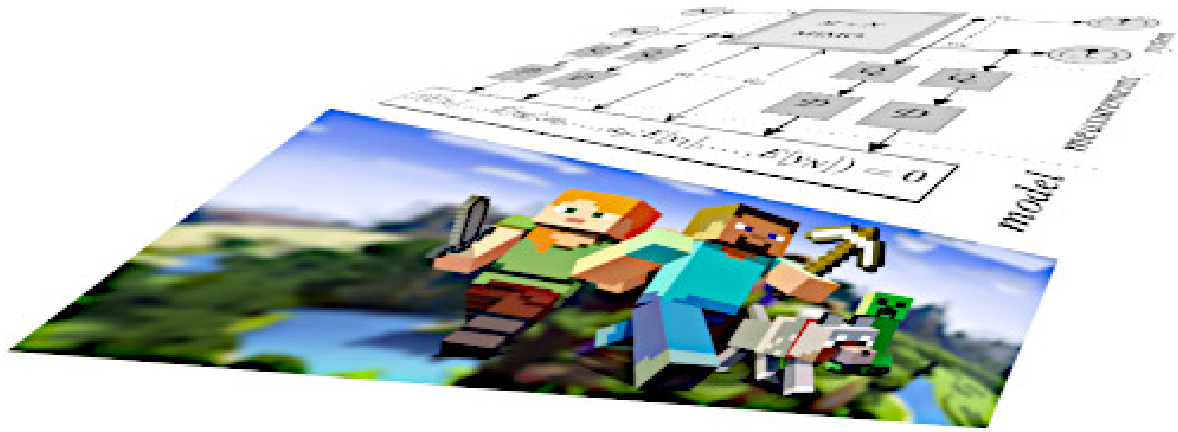
Minecraft of System Modeling



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About Me

Pavel Loskot joined the ZJU-UIUC Institute as Associate Professor in January 2021. He received his PhD degree in Wireless Communications from the University of Alberta in Canada, and the MSc and BSc degrees in Radioelectronics and Biomedical Electronics, respectively, from the Czech Technical University of Prague. He is the Senior Member of the IEEE, Fellow of the HEA in the UK, and the Recognized Research Supervisor of the UKCGE. He was elected the IARIA 2025 Fellow.

In the past nearly 30 years, he was involved in numerous industrial and academic collaborative projects in the Czech Republic, Finland, Canada, the UK, Turkey, and in China. These projects concerned wireless and optical telecommunication networks, and also genetic regulatory circuits, air transport services and renewable energy systems. This experience allowed him to truly understand the interdisciplinary workings, and crossing the disciplines boundaries.

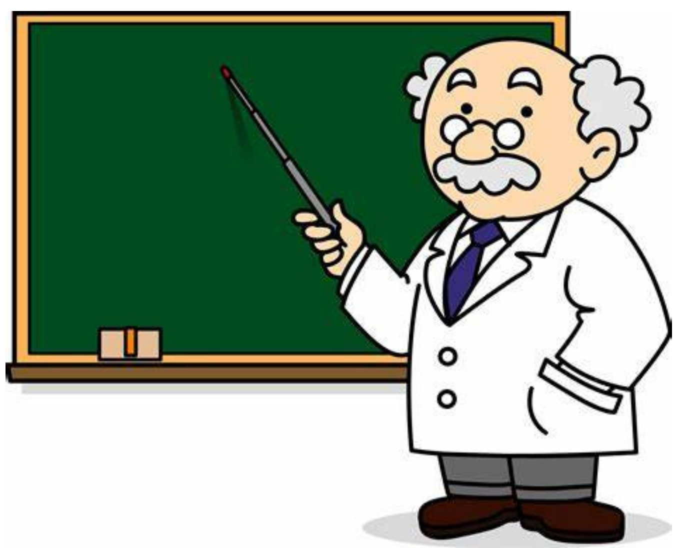
His current research focuses on mathematical and probabilistic modeling, statistical signal processing and classical machine learning for multi-sensor data in biomedicine, computational molecular biology, and wireless communications.

Objectives

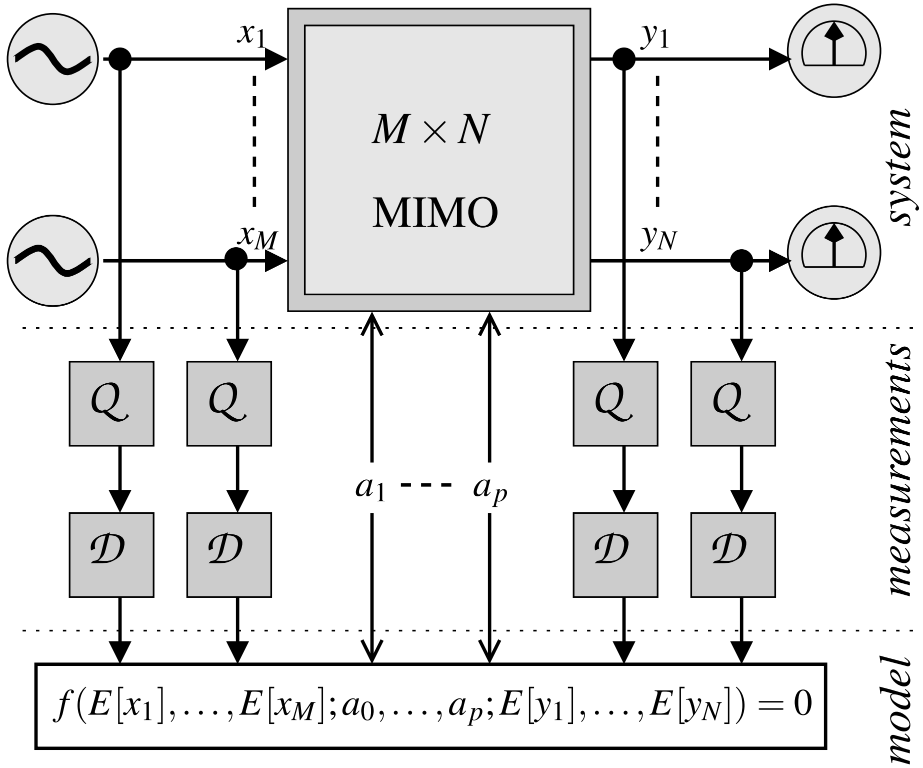
- estimate model parameters from quantized noisy inputs and outputs
  → this is a standard problem of model identification
  → quantization is akin to Minecraft modeling
- assumptions
  → system model inputs and outputs are noisy constants
  → system model is linearized
- compare the variances of ML estimators with quantized observations
  → uniform and binary quantization
  → unquantized measurements

Outline

- System model
- Parameter estimation
- Maximum-likelihood estimator
- Numerical examples



System Model



Quantization

- implicit
  → measuring equipment with limited resolution
- explicit
  → reduce storage and computing requirements

Measurements

$$\mathrm{E}[x] \neq \mathrm{Av}[\mathrm{E}[x]] \neq \mathrm{Av}[x] \quad \Leftrightarrow \quad \mathrm{non-ergodic\ \&\ non-stationary}$$

System Model (2)

Linear SISO model

$$y = a_0 + \sum_{i=1}^{p} a_i \phi_i(x), \quad \left\langle \phi_i, \phi_j \right\rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Measurements

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & \phi_1(x_1) & \cdots & \phi_p(x_1) \\ \vdots & \vdots & & \vdots \\ 1 & \phi_1(x_n) & \cdots & \phi_p(x_n) \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{bmatrix}$$

$$\boldsymbol{y} = \boldsymbol{\Phi}(\boldsymbol{x}) \cdot \boldsymbol{a}$$

Linearization

$$\phi_i(x) \doteq \phi_i(x_0) + \dot{\phi}_i(x_o)(x - x_0)$$

$$\Rightarrow \quad y = a_0 + \sum_{i=1}^{p} a_i (A_i x + B_i) \quad \Rightarrow \quad \begin{cases} A_i = \dot{\phi}_i(x_0) \\ B_i = \phi_i(x_0) - \dot{\phi}_i(x_0) x_0 \end{cases}$$

System Model (3)

Uniform quantization

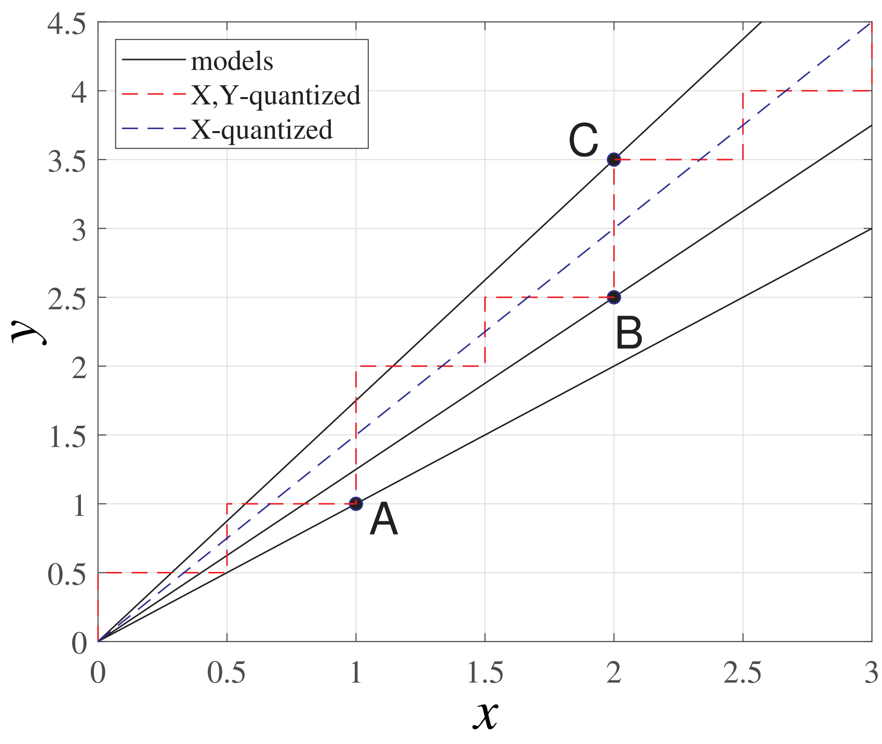
$$\check{x} = Q(x) = \left\lfloor \frac{x - \Delta/2}{\Delta} \right\rfloor + 1 \in \mathbb{Z}$$

$$\Delta(\check{x} - 1/2) \leq x < \Delta(\check{x} + 1/2) \quad \text{(quantization error)}$$

Binary quantization

$$\check{x} = Q_2(x) = \mathrm{sign}(x) \in \begin{cases} +1, \\ -1 \end{cases}$$

Example: $y = \frac{3}{2}x$



Parameter Estimation

Measurements $i = 1, 2, \ldots, n$

$$\begin{aligned} y_i &= \bar{y} + \epsilon_{yi} \\ x_i &= \bar{x} + \epsilon_{xi} \end{aligned} \quad \Rightarrow \quad \mathrm{E}[x_i y_i] = \bar{x}\bar{y} + \underbrace{\mathrm{E}\big[\epsilon_{xi}\epsilon_{yi}\big]}_{\neq 0}$$

Linearized model

$$\boldsymbol{y} = \left[\mathbf{1}_{(n,1)} \mid \bar{\boldsymbol{\Phi}}(\bar{x}) + \boldsymbol{\epsilon}_x \cdot \dot{\boldsymbol{\phi}}^T(\bar{x})\right] \cdot \boldsymbol{a}$$

The LS model fitting

$$\mathrm{LS}(a_0, \boldsymbol{a}) = \sum_{i=1}^{n} \left(y_i - a_0 - \left(\dot{\boldsymbol{\phi}}\,\epsilon_{xi} + \boldsymbol{\phi}\right)^T \cdot \boldsymbol{a}\right)^2 \quad \Rightarrow \quad \begin{aligned} \tfrac{\partial}{\partial a_0}\mathrm{LS}(\hat{a}_0, \hat{\boldsymbol{a}}) &= 0 \\ \tfrac{\partial}{\partial \boldsymbol{a}}\mathrm{LS}(\hat{a}_0, \hat{\boldsymbol{a}}) &= \mathbf{0} \end{aligned}$$

$$\Rightarrow \quad \left\{ \begin{aligned} \hat{a}_0 &= \mathrm{Av}[y_i] - \left(\dot{\boldsymbol{\phi}}\mathrm{Av}[\epsilon_{xi}] + \boldsymbol{\phi}\right)^T \hat{\boldsymbol{a}} \\ \hat{\boldsymbol{a}} &= \left(\dot{\boldsymbol{\phi}}\dot{\boldsymbol{\phi}}^T\right)^{-1} \dot{\boldsymbol{\phi}} \frac{\mathrm{Av}\big[\epsilon_{xi}\epsilon_{yi}\big]}{\mathrm{Av}\big[(\epsilon_{xi} - \bar{\epsilon}_x)^2\big]} + \underbrace{\left(\dot{\boldsymbol{\phi}}\dot{\boldsymbol{\phi}}^T\right)^{-1} \boldsymbol{\phi} \frac{\mathrm{Av}\big[\epsilon_{yi}\big]}{\mathrm{Av}\big[(\epsilon_{xi} - \bar{\epsilon}_x)^2\big]}}_{=0 \text{ for } n \gg 1} \end{aligned} \right.$$

Parameter Estimation (2)

A SISO case

$$\hat{a}_0 = \bar{y} - \hat{a}_1 \bar{x}\ , \quad \hat{a}_1 = \frac{\mathrm{Av}[(y_i - \bar{y})(x_i - \bar{x})]}{\mathrm{Av}[(x_i - \bar{x})^2]}$$

- if the inputs and outputs are noisy constants, and $n \gg 1$

$$\mathrm{MSE}(\hat{a}_0, \hat{a}_1) = \mathrm{E}\left[\epsilon_{yi}^2\right] - \frac{\mathrm{E}\left[\epsilon_{xi}\epsilon_{yi}\right]^2}{\mathrm{E}\left[\epsilon_{xi}^2\right]}$$

Alternative strategy for model identification

1. accurately estimate the constant inputs and outputs
2. linearize the model about these estimates
3. invert the model: $\boldsymbol{a} = \boldsymbol{\Phi}^{-1}(\boldsymbol{x}) \cdot \boldsymbol{y}$

Estimating noisy constant

- minimum variance unbiased (MVUB) estimator
  $\rightarrow$ best linear unbiased (BLUE) estimator
- LS estimator
  $\rightarrow$ performs poorly
- maximum-likelihood estimator

Maximum-Likelihood Estimator

Strategy

- assume SISO model
- measurement noises are AWGN, and measurements are independent
- measurements are quantized (uniform or binary)
- better to consider the log-likelihood
- linearize Gaussian Q-function and its derivative

Specifically

$$\frac{\partial}{\partial \bar{x}} \log \Pr(\{\check{x}_i\}_i) = -\frac{1}{\sigma} \sum_{i=1}^{n} \frac{\dot{Q}\left(\frac{\Delta(\check{x}_i - 1/2) - \bar{x}}{\sigma}\right) - \dot{Q}\left(\frac{\Delta(\check{x}_i + 1/2) - \bar{x}}{\sigma}\right)}{Q\left(\frac{\Delta(\check{x}_i - 1/2) - \bar{x}}{\sigma}\right) - Q\left(\frac{\Delta(\check{x}_i + 1/2) - \bar{x}}{\sigma}\right)}$$

$$Q(x) \approx Q(x_0) - \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x_0^2/2}(x - x_0)$$

$$\dot{Q}(x) \approx \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x_0^2/2}(x_0 x - x_0^2 - 1)$$

Maximum-Likelihood Estimator (2)

ML estimators

$$\hat{\hat{x}} = \Delta \frac{1}{n} \sum_{i=1}^{n} \check{x}_i \qquad \text{(uniform quantization)}$$

$$\hat{\hat{x}} = \sigma \sqrt{\frac{\pi}{2}} \frac{1}{n} \sum_{i=1}^{n} \check{x}_i \qquad \text{(binary quantization)}$$

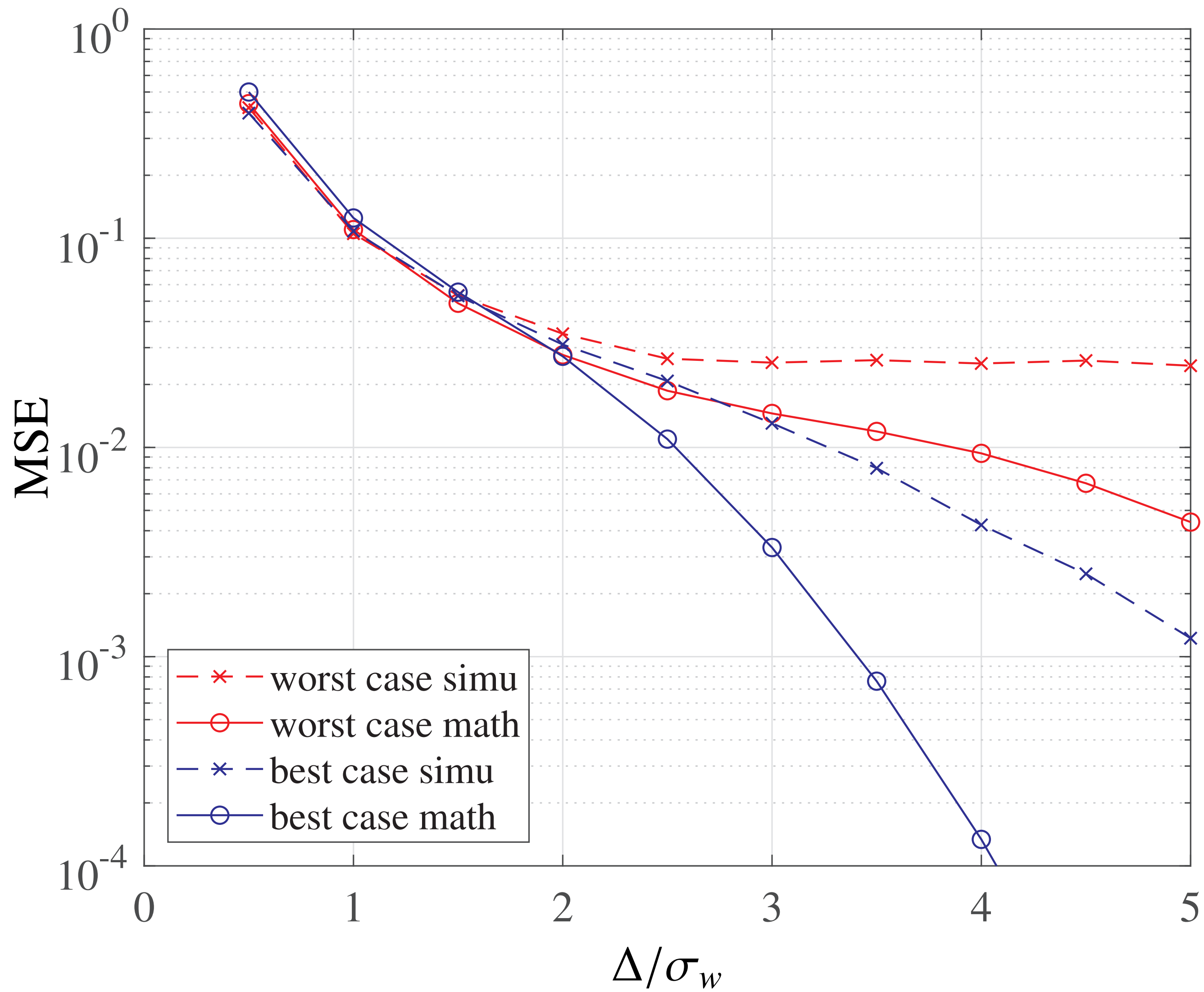
$$\hat{\hat{x}} = \frac{1}{n} \sum_{i=1}^{n} x_i \qquad \text{(no quantization)}$$

Variances

- derived in the paper
- for uniform quantization
  → best case: no quantization error
  → worst case: maximal quantization error
- approximations
  → measurement noise comparable or smaller than quantization noise

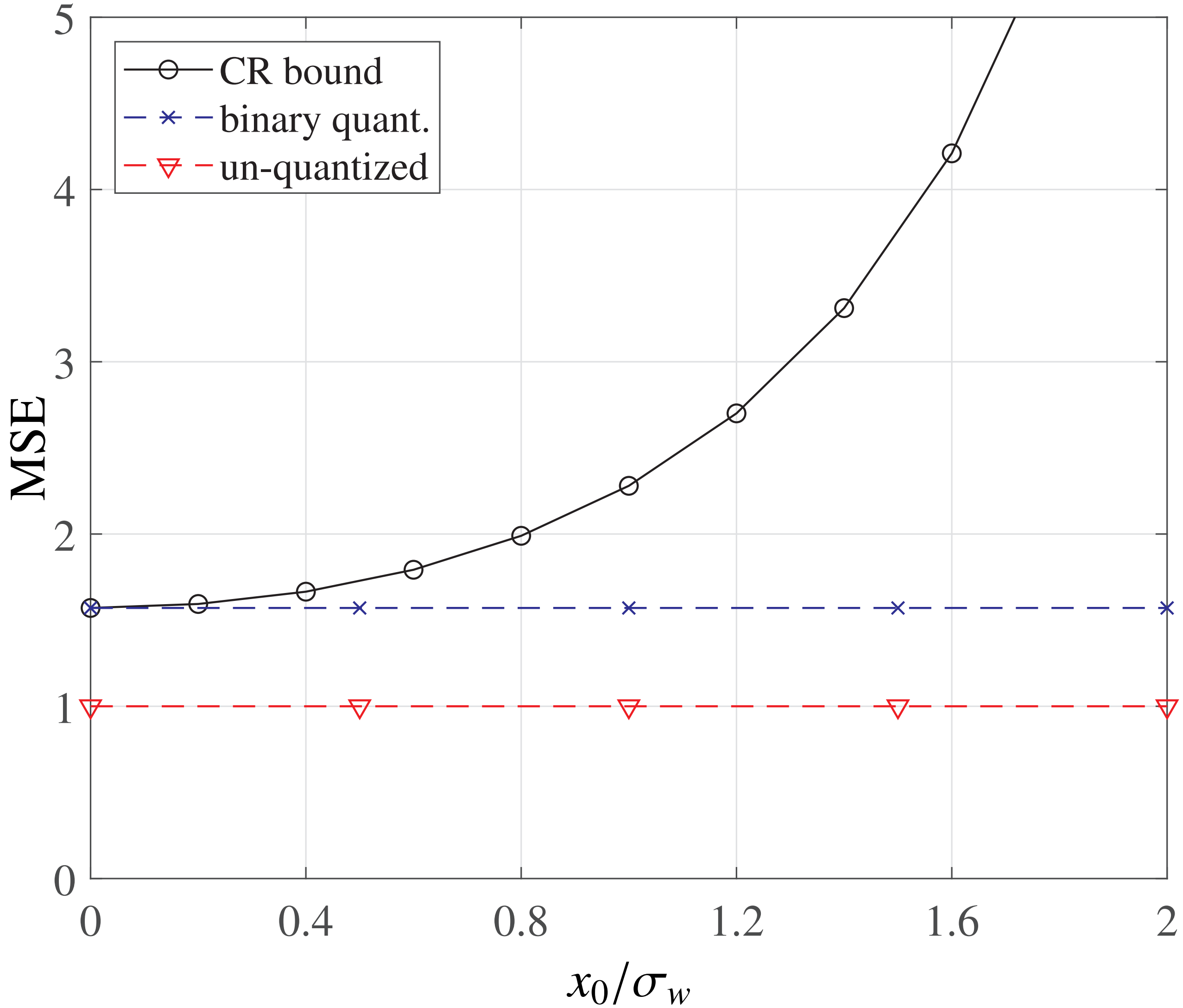
Numerical Examples

Uniform quantization



Numerical Examples (2)

Binary quantization



Take-Home Messages

System identification

- if inputs and outputs are static (noisy constants)
- LS fitting performs poorly
- much better is to estimate inputs and outputs from multiple measurements
  $\rightarrow$ different estimators available
- then linearize and invert the model

Key observations

- with static inputs, system model can be readily linearized
- quantization noise can be neglected if comparable to measurement noise
- if this is not the case
  $\rightarrow$ the estimators are no longer unbiased and consistent

Direct implications

- supervised machine learning
- physical laws
  $\rightarrow$ Schrödinger and Maxwell's equations are linear
- Minecraft perception of reality

Thank you!

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