A Control Framework for Direct Adaptive State & Input Matrix Estimation

With Known Inputs for LTI Dynamic Systems

Adaptive 2025

By: *Kevin Fuentes*, Dr. Mark Balas, & Dr. James Hubbard
April 06-10, 2025

Kevin Fuentes

- Doctoral Candidate in the Mechanical Engineering Department at Texas A&M University.
  - Expected to defend my thesis in 2025.

- Research consists of developing adaptive control schemes for plant and state estimation accounting for model uncertainty or changes in the physical dynamics.

- Work is being supervised by Dr. Mark Balas & Dr. James Hubbard.

Contact Email: kfuentes@tamu.edu

Overview

- Motivation
- Background
- Adaptive State and Input Matrix Estimator
- Illustrative Example
- Conclusion

Motivation

- Many dynamic systems experience performance degradation with use or age.
  - Altering physical dynamics or constitutive constants.

- Not accounting for these changes in the model can lead to unreliable state information.
  - Complications can arise if the true-physical system movement is dependent on estimated state information.

Model vs. Physical Dynamics

- A model is defined by governing equation(s) of motion (**EOM**).
- A model could predict dynamics of a true-physical systems under a set of assumptions and constrains.
    - Example: Euler-Bernoulli Beam Assumes…
        - Small Deflections, low frequency excitation, and no rotary inertia.
- However, the **model is not a physical system.**

Let the model and physical systems be described as Linear Time Invariant (LTI).

LTI System

- Any LTI system can be described in state space:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

  - $A \rightarrow$ Plant
  - $B \rightarrow$ Input Matrix
  - $C \rightarrow$ Output Matrix
  - $u \rightarrow$ Input
  - $x \rightarrow$ Internal State
  - $y \rightarrow$ External (Output) State

- System satisfies superposition and scaling.

If you have a “good” LTI system, internal states can be estimated.

Luenberger (State) Observer

- Luenberger (State) Observers dates back to the 1970s [Luenberger, 1971].
- Requires minimal uncertainty about plant dynamics.
- Plant must be Observable (A,C).

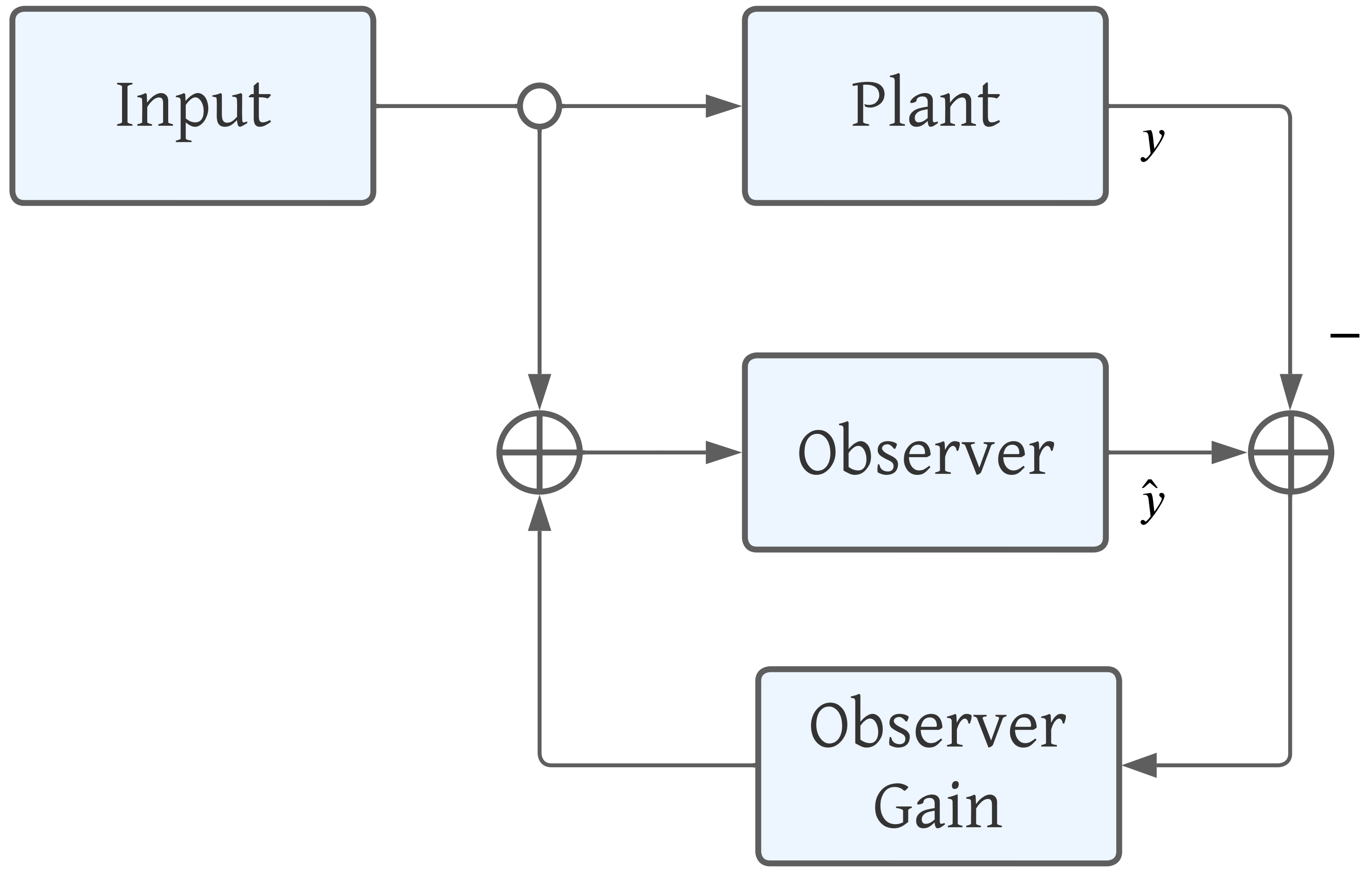


Figure 1: Generalized Luenberger Observer Control Diagram.

Kalman Filters

- Kalman Filters assumes noise exist in the system.
  - Noise assumes to follow a gaussian distribution with zero mean [Kalman, 1960].
- As with Luenberger, internal states are estimated using an iterative process.
- Observer Gain is selected base on estimated and measured state confidence.

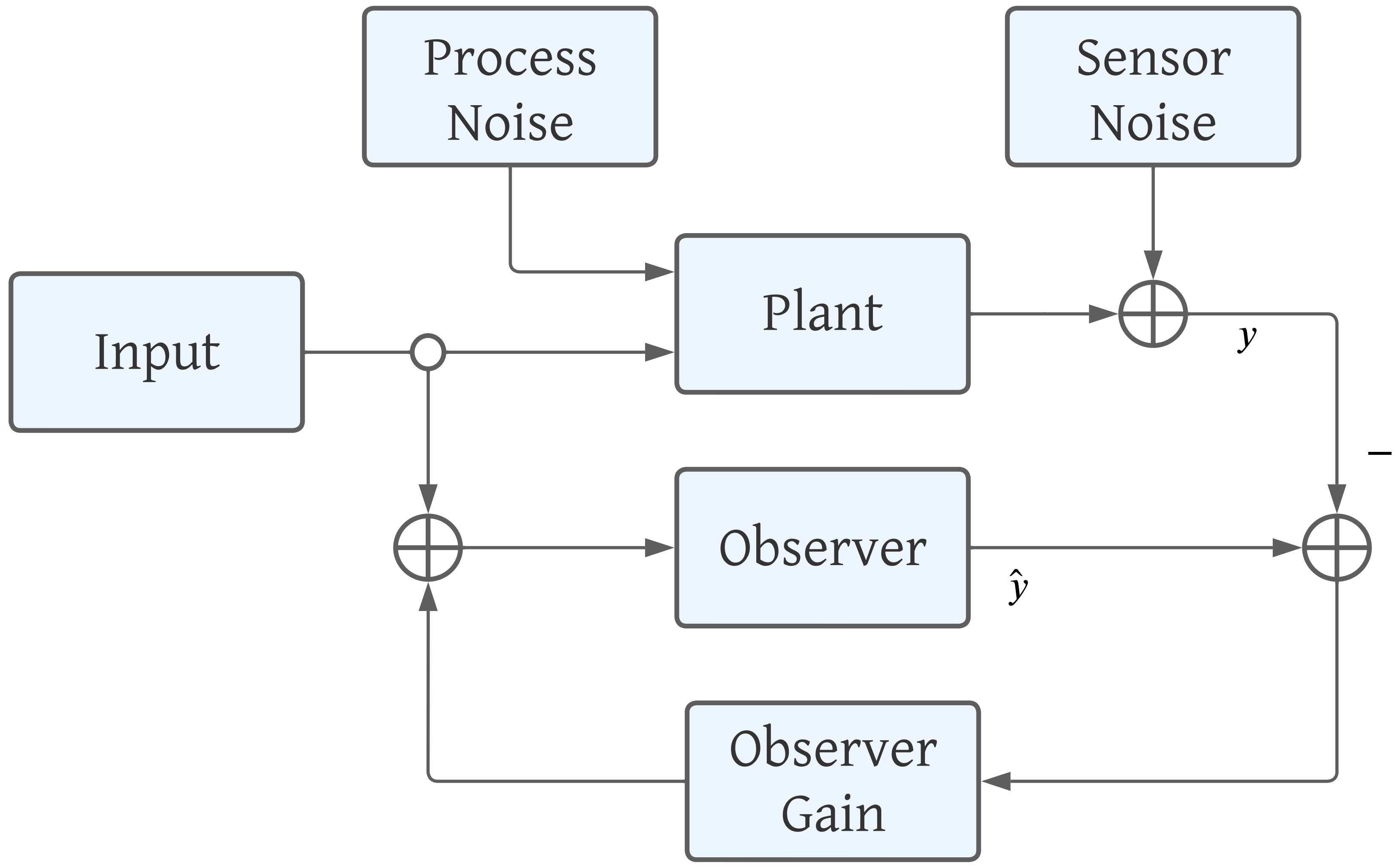


Figure 2: Generalized Kalman Filter Diagram.

Background: Model Uncertainty

Model Uncertainty

- Model uncertainty is caused by…
  - **All dynamics are not accounted for inside the EOM.**
  - **Not knowing the correct constitutive relations.**
  - Having process/sensor noise.
- Robustness techniques exist to limit the effects of model uncertainty:
  - H_∞ Synthesis [Nagpal, 1991]
  - μ Synthesis [Doyle, 1987]

The proposed control scheme can account for model uncertainty

..if uncertainty is inside the dimensions of the model input matrix and plant.

More importantly, the proposed control scheme deals with model uncertainty when a system experiences a “significant” health status change.

What does accounting for "significant" health status change mean?

- With regard to system dynamics, if the "significant" health change can be defined within plant estimation constraint:

$$B \in \text{sp}\{B_m L_{1*}\} \ni B = B_m L_{1*}$$

  - $B \rightarrow$ True-Physical Input Matrix
  - $B_m \rightarrow$ Initial Input Matrix Model
  - $B_m L_{1*} \rightarrow$ Input Matrix Correction Term

$$A \in \text{sp}\{A_m, B_m L_{2*} C\} \ni A = A_m + B_m L_{2*} C$$

  - $A \rightarrow$ True-Physical Plant
  - $A_m \rightarrow$ Initial Plant Model
  - $B_m L_{2*} C \rightarrow$ Plant Correction Term

- The proposed control scheme will update in time to reflect these changes under specific assumptions and constraints.
- $\{A_m, B_m, C\}$ are known.
- $\{A_m, A\}$ are stable.

Inspiration for Model Updating

A Modal Approach to the Space-Time Dynamics of Cognitive Biomarkers

- An adaptive unknown input approach to brain wave EEG Estimation
  - Griffith, Balas, & Hubbard proposed an open-loop coupled approached to input and state estimation [Griffith 2023].

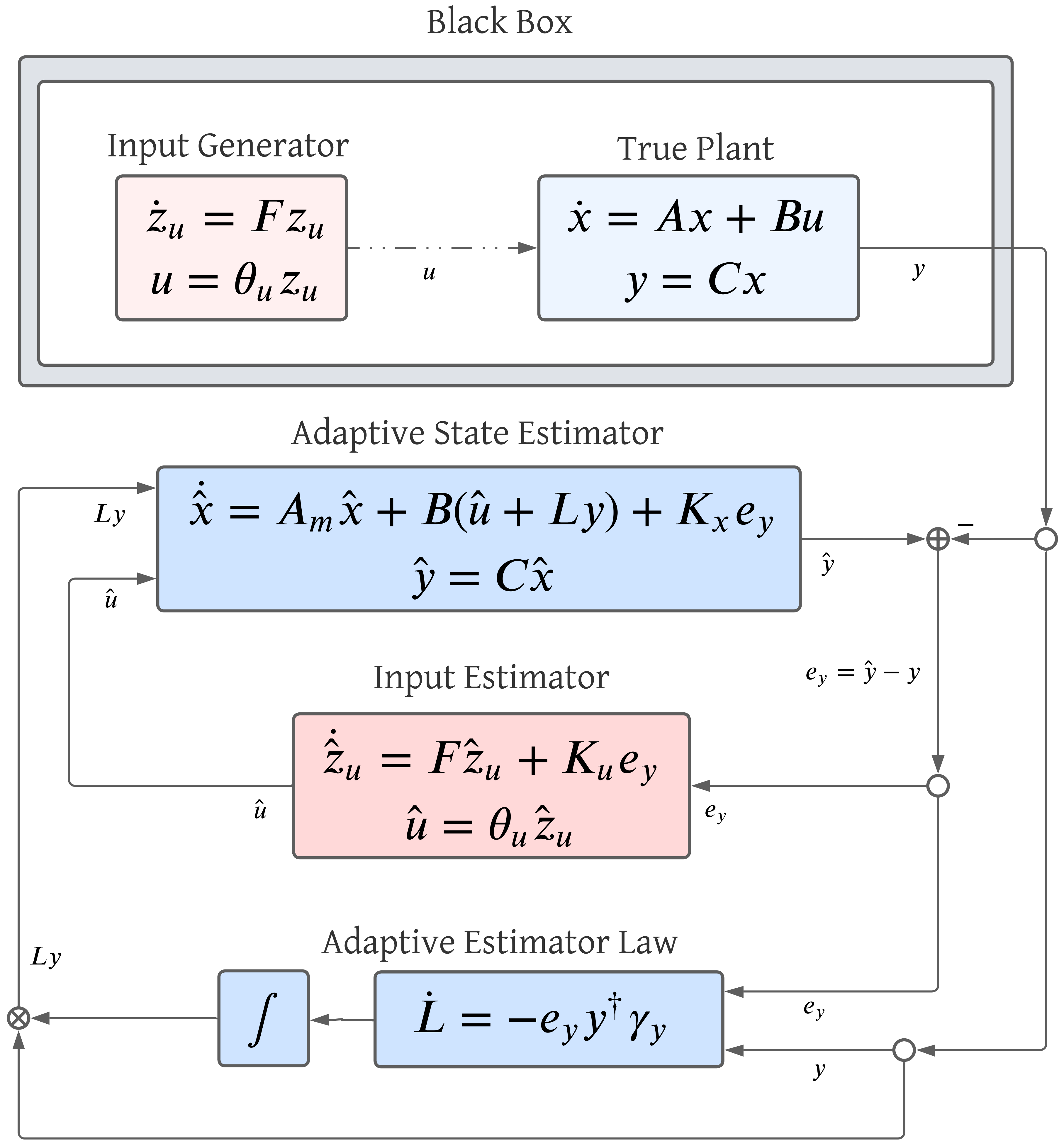
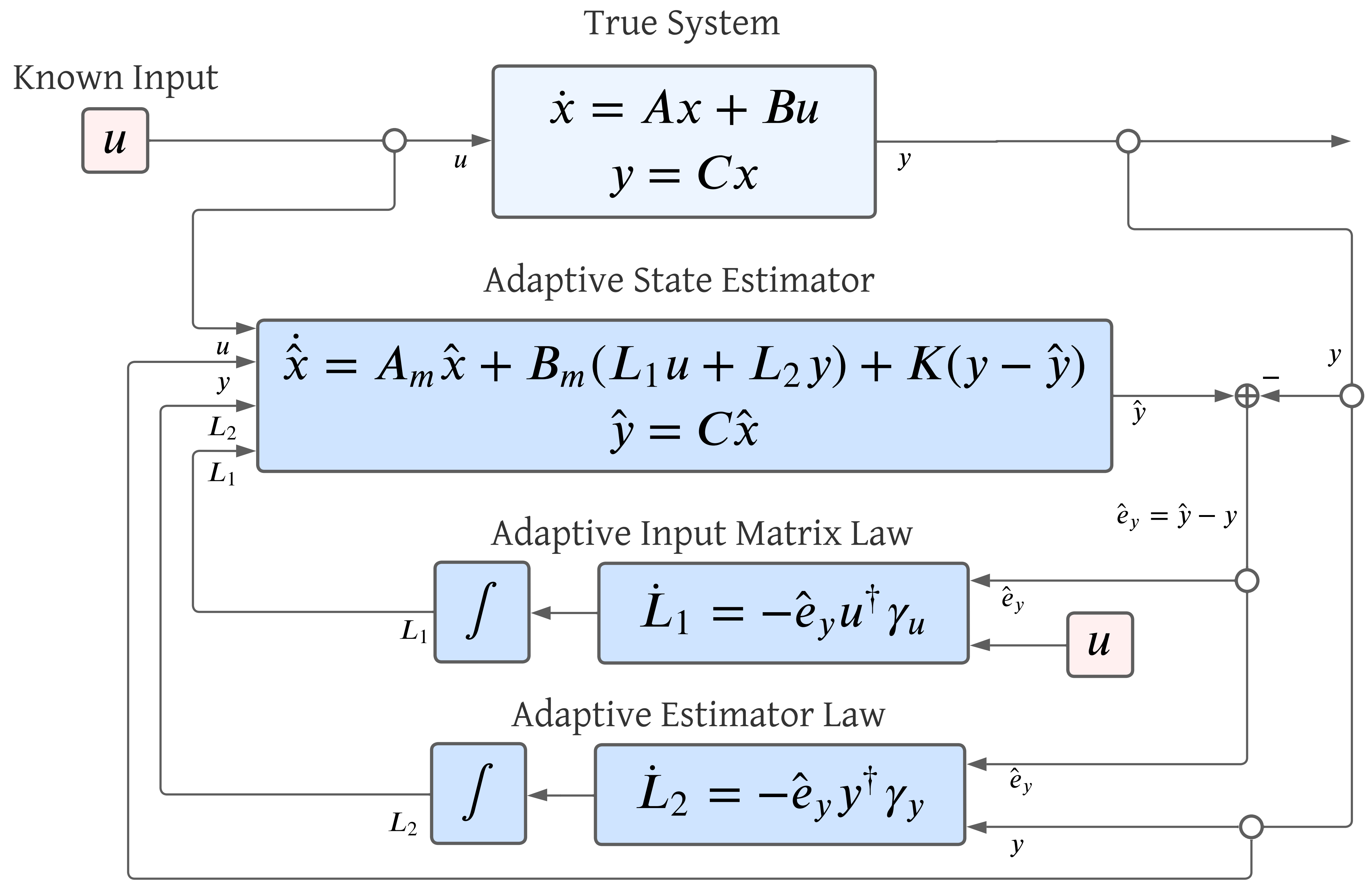


Figure 3: Generalized unknown input estimator for brain wave estimation.

What if we can send information via an input to the true system?

A Control Framework for Direct Adaptive State & Input Matrix Estimation

with Known Inputs for LTI Dynamic Systems



Decomposition of the True System's Input Matrix and Plant

- Let the true system's input matrix be composed of the model input matrix (B_m) and the correction matrix (L_{1*})

$$B \equiv B_m L_{1*}$$

- The true plant be composed of the model plant (A_m) and the correction matrix (L_{2*})

$$A \equiv A_m + B_m L_{2*} C$$

- Can we determine $\{L_{1*}, L_{2*}\} \ni$

$$L(t) = \Delta L + L_* \xrightarrow[t \to \infty]{} L(t) = L_*$$

where ΔL is the variance in L?

Structure of 'True' Plant

- This structure of the 'true' plant (A) can be derived $\ni A \equiv A_m + B_m L_{2*} C$ from:

$$\begin{aligned} Ax &= A_m x + B_m L_{2*} y \\ &= A_m x + B_m L_{2*} C x \;\; \rightarrow A = A_m + B_m L_{2*} C. \\ &= (A_m + B_m L_{2*} C) x \end{aligned}$$

- **Why this form?**
    - A_m gives initial plant structure.
    - The input matrix (B_m) actuates the system.
    - System Output (y) has state information of the true system.

Adaptive State Estimator

- Since the true plant and input matrix is unknown and state information is often inaccessible.

- An observer-estimator using the reference model plant (A_m) can be made:

$$\text{Adaptive State Estimator} \begin{cases} \dot{\hat{x}} = A_m\hat{x} + B_m(L_1u + L_2y) \\ \hat{y} = C\hat{x} \end{cases},$$

where input (u) can be any bounded-continuous waveform.

Error Dynamics

- Allowing the true input matrix (B) and plant (A) be decomposed such that:

$$\begin{cases} B = B_m L_{1*} \\ A = A_m + B_m L_{2*} C \end{cases}$$

- Results in the **error dynamics** can be written as:

$$\begin{cases} \dot{e}_x = A_m e_x + B_m(\underbrace{\Delta L_1 u}_{=w_u} + \underbrace{\Delta L_2 y}_{=w_y}) \\ \hat{e}_y = C e_x \end{cases}.$$

- No guarantee that $e_x \xrightarrow[t\to\infty]{} 0$ because of the residual terms $\{B_m \Delta L_1 u, B_m \Delta L_2 y\}$ in the error equation.

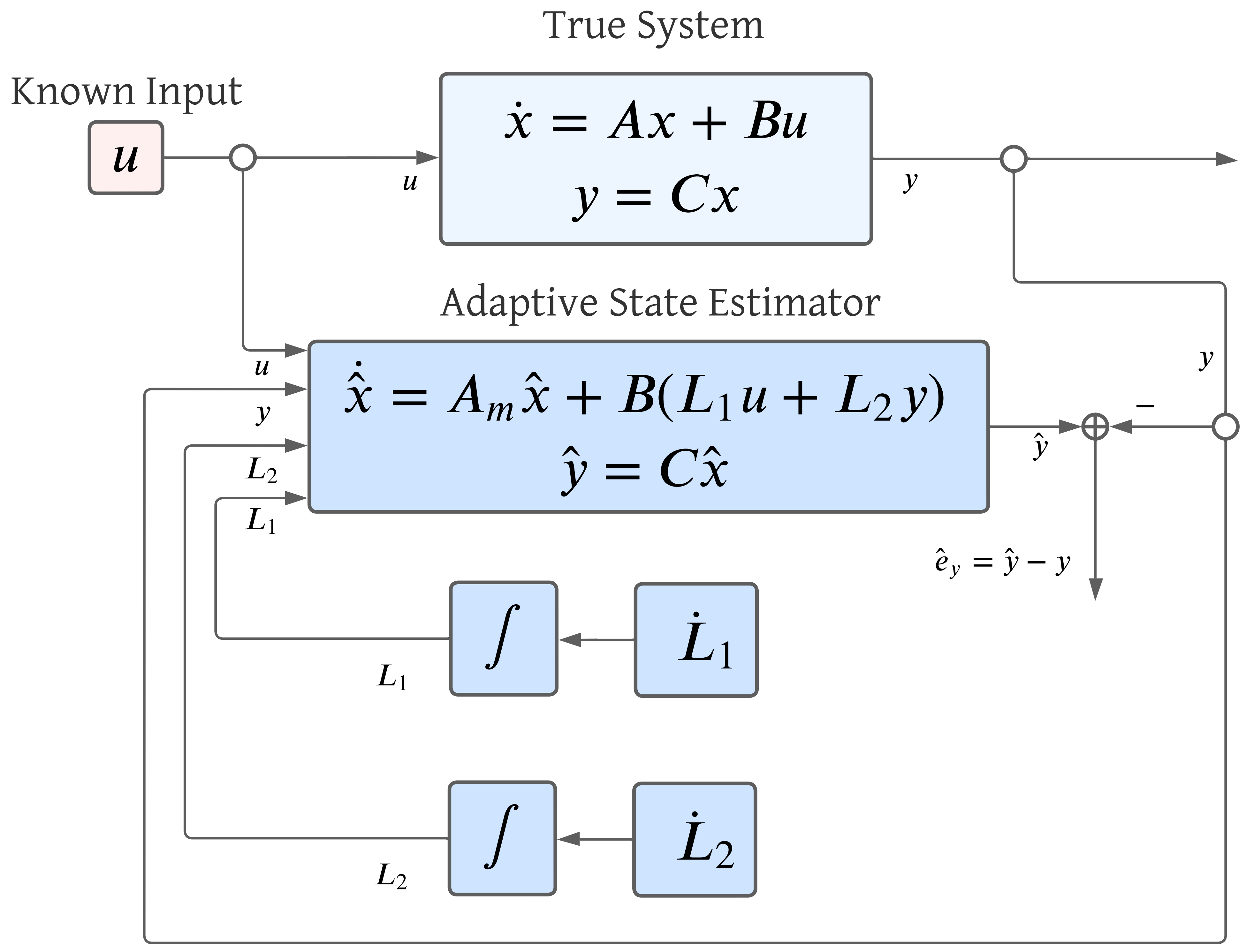


Figure 4: Partial Adaptive State Estimator.

Lyapunov Analysis

Lyapunov Stability

- Why do we care about Lyapunov Stability?
  - Lyapunov argument considered dynamic systems in terms of energy-like functions.
  - In this case, we are considering the energy rate of change for the error state to guarantee $e_x \xrightarrow[t \to \infty]{} 0$.
    - If error energy can be dissipated, estimated state converges to the true state.

Lyapunov Proof Results

- Lyapunov analysis results in the **adaptive estimation law**:

$$\Delta\dot{L}_1 = \dot{L}_1 = -\,\hat{e}_y u^{\dagger}\gamma_u; \;\; \gamma_u > 0.$$

$$\Delta\dot{L}_2 = \dot{L}_2 = -\,\hat{e}_y y^{\dagger}\gamma_y; \;\; \gamma_y > 0.$$

- Proof guarantees $e_x \xrightarrow[t\to\infty]{} 0$ and $\hat{e}_y \xrightarrow[t\to\infty]{} 0$ asymptotically.
- $\{\Delta L_1, \Delta L_2\}$ are guaranteed to bounded.
- No guarantee $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$.
  - If $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$ **numerically**, the dynamics of the true plant can be been captured. [Fuentes, 2025].

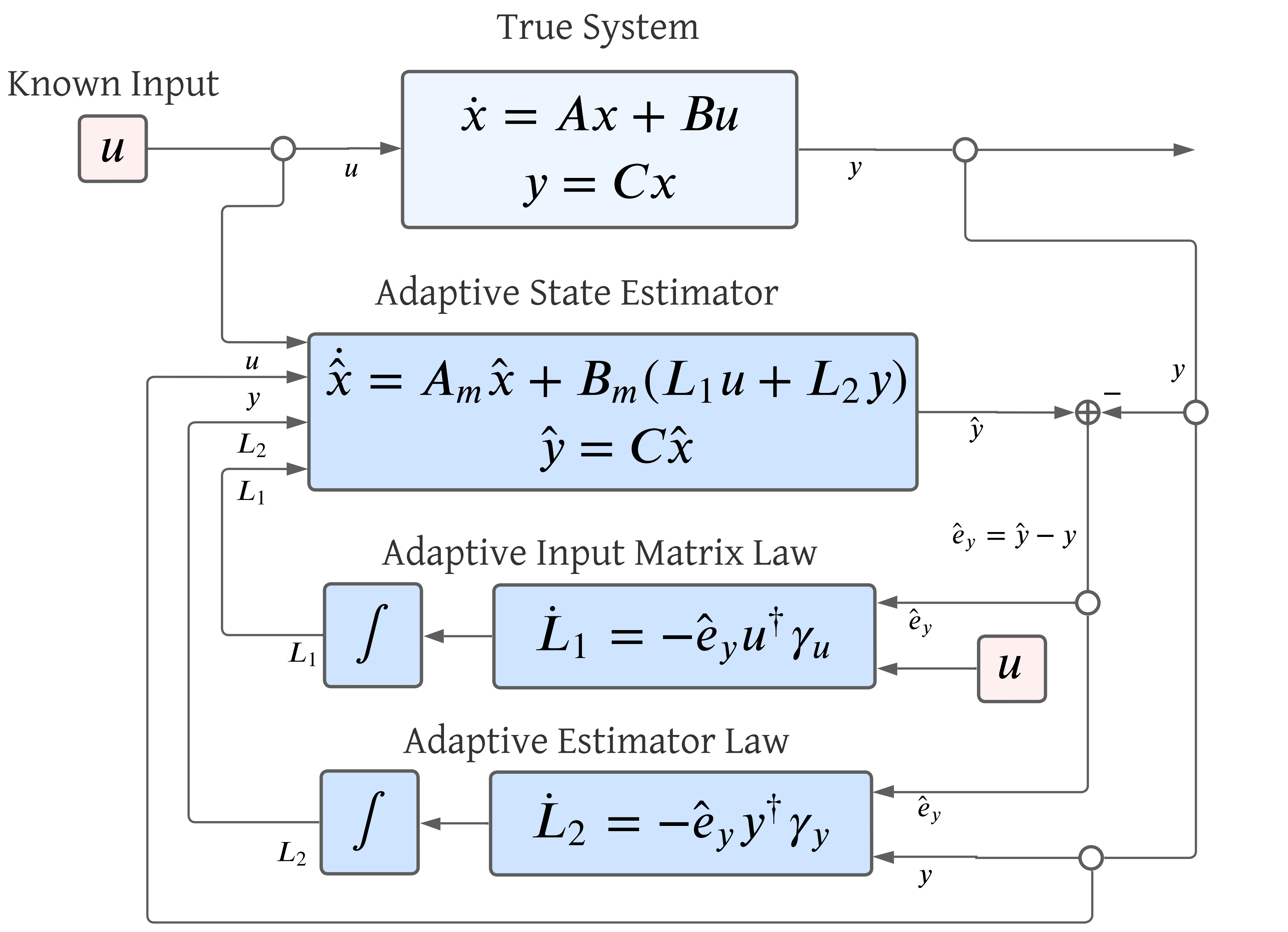


Figure 5: Adaptive State and Input Matrix Estimator.

$(\cdot)^{\dagger}$ = conjugate transpose

WLOG - Use of Fixed Gains

- Given the following **error system**:

$$\begin{cases} \dot{e}_x = (A_m - KC)e_x + B_m(w_u + w_y) \\ \hat{e}_y = Ce_x \end{cases}.$$

- Lyapunov analysis results in the **adaptive estimation law**:

$$\Delta\dot{L}_1 = \dot{L}_1 = -\,\hat{e}_y u^{\dagger}\gamma_u;\;\; \gamma_u > 0.$$

$$\Delta\dot{L}_2 = \dot{L}_2 = -\,\hat{e}_y y^{\dagger}\gamma_y;\;\; \gamma_y > 0.$$

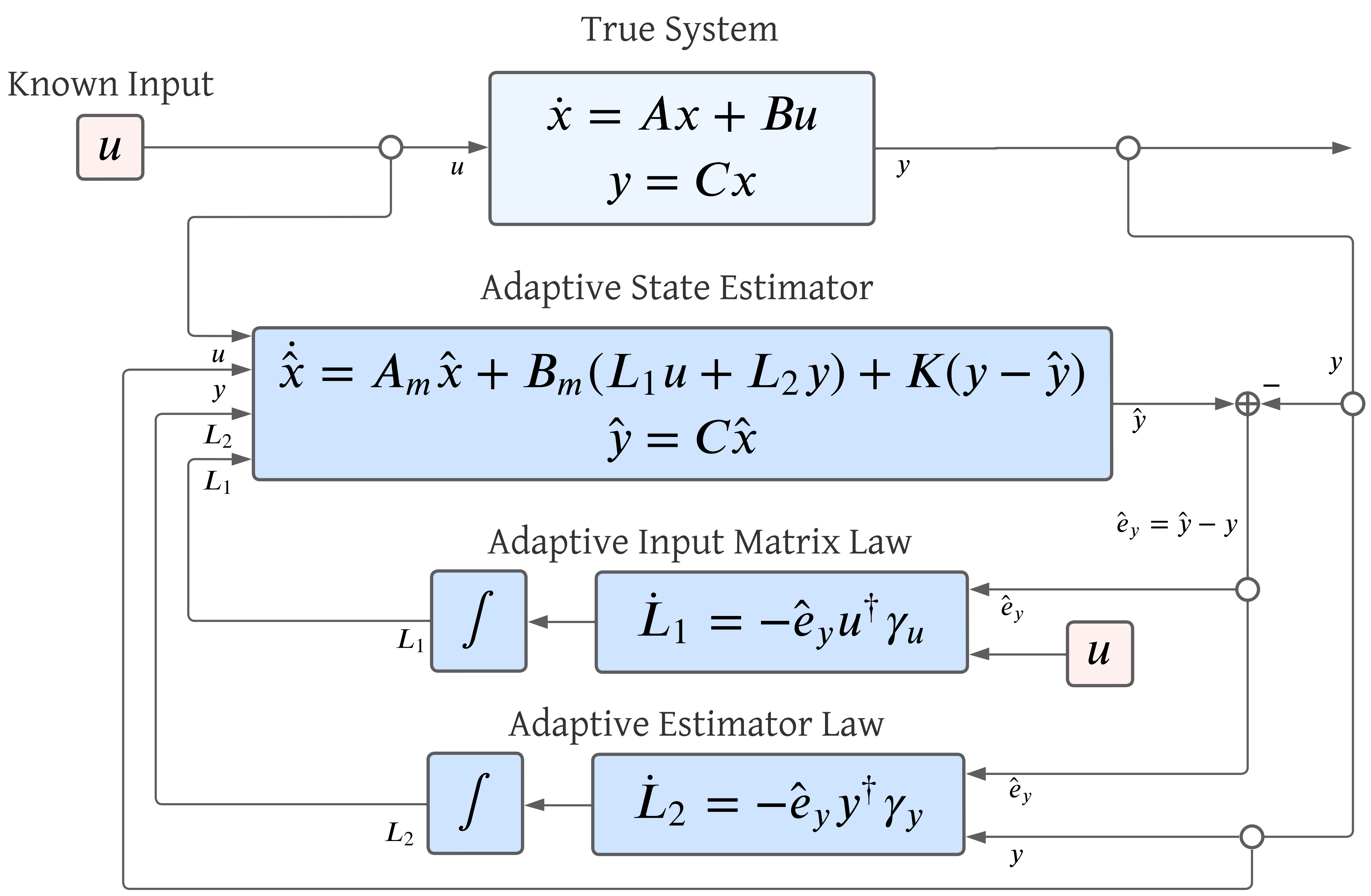


Figure 6: Adaptive State Estimator using a Fixed Gain (K).

Illustrative Example

Defining the Dynamics

- With appropriate modeling, let a reference model and plant (A_m) exist ∋

$$\text{Reference Model} \begin{cases} \dot{x}_m = A_m x_m + B_m u \\ y_m = C x_m \end{cases}; A_m = \begin{bmatrix} -7 & 2 & 4 \\ -2 & -1 & 2 \\ -2 & 2 & -1 \end{bmatrix}; B_m = \begin{bmatrix} 0 \\ 0.7 \\ 2 \end{bmatrix}; C = [0.5 \quad 0 \quad 1].$$

- For the proposed control approach to be viable, allow: $B \in \text{sp}\{B_m L_{1*}\} \ni B = B_m L_{1*}$ & $A \in \text{sp}\{A_m, B_m L_{2*} C\} \ni A = A_m + B_m L_{2*} C$:

$$\text{True System} \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}; A = A_m + BL_{2*}C = \begin{bmatrix} -7 & 2 & 4 \\ -3.75 & -1 & -1.5 \\ -7 & 2 & -11 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1.4 \\ 4 \end{bmatrix}; C = [0.5 \quad 0 \quad 1].$$

System Response

- Giving a unit input response to the True and Reference model, notice the significant difference in output response.

True System $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

Reference Model $\begin{cases} \dot{x}_m = A_m x_m + B_m u \\ y_m = C x_m \end{cases}$

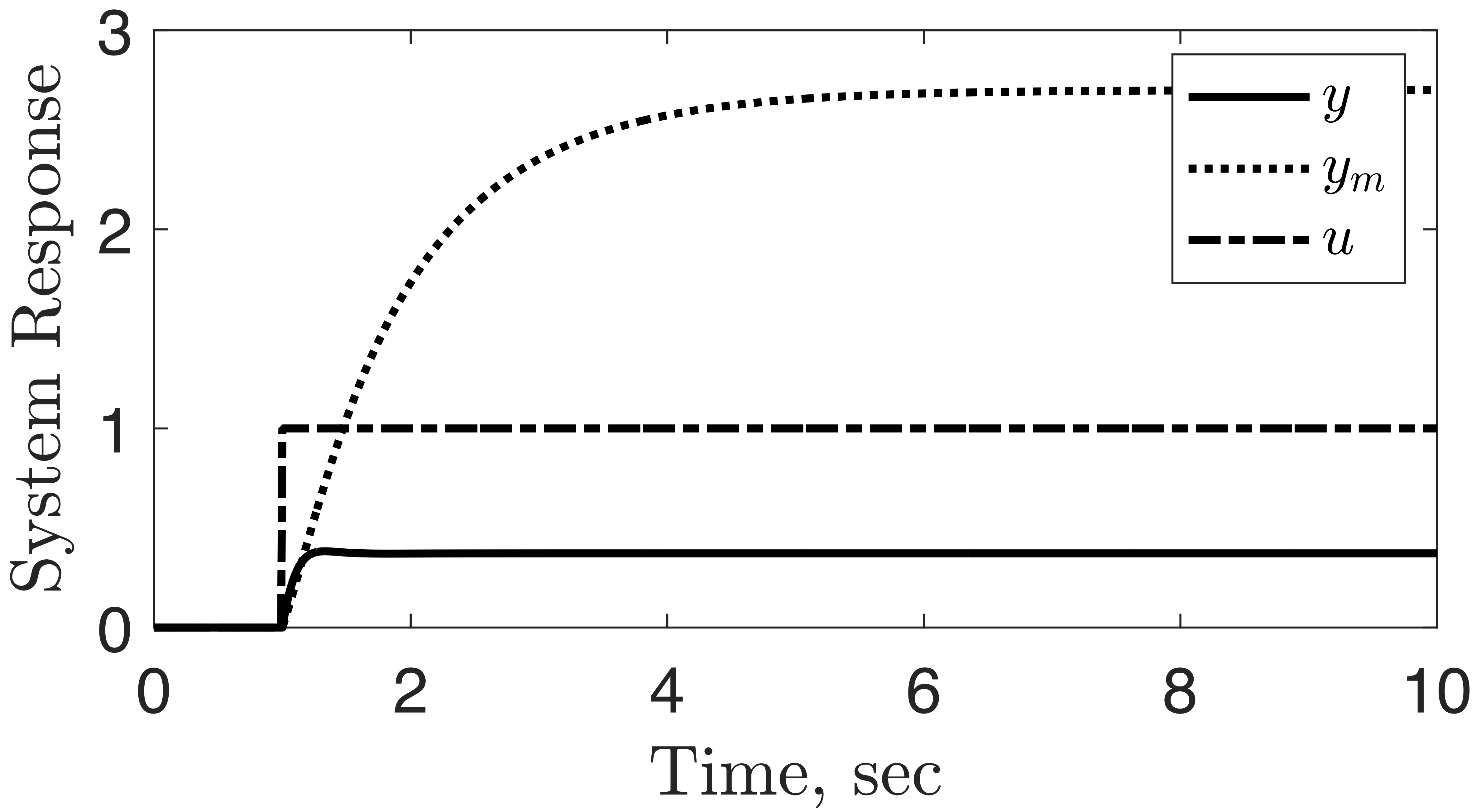


Figure 7: Output response for the true model (y) and reference model (y_m) given a unit step input (u).

Selecting Input

- Any bounded-continuous input (u) can be injected into the True and Estimator systems, proof guarantees $e_x \xrightarrow[t\to\infty]{} 0$ and $\hat{e}_y \xrightarrow[t\to\infty]{} 0$ asymptotically.

- Lets define the known input as:

$$u = 2 + \sin(2t)$$

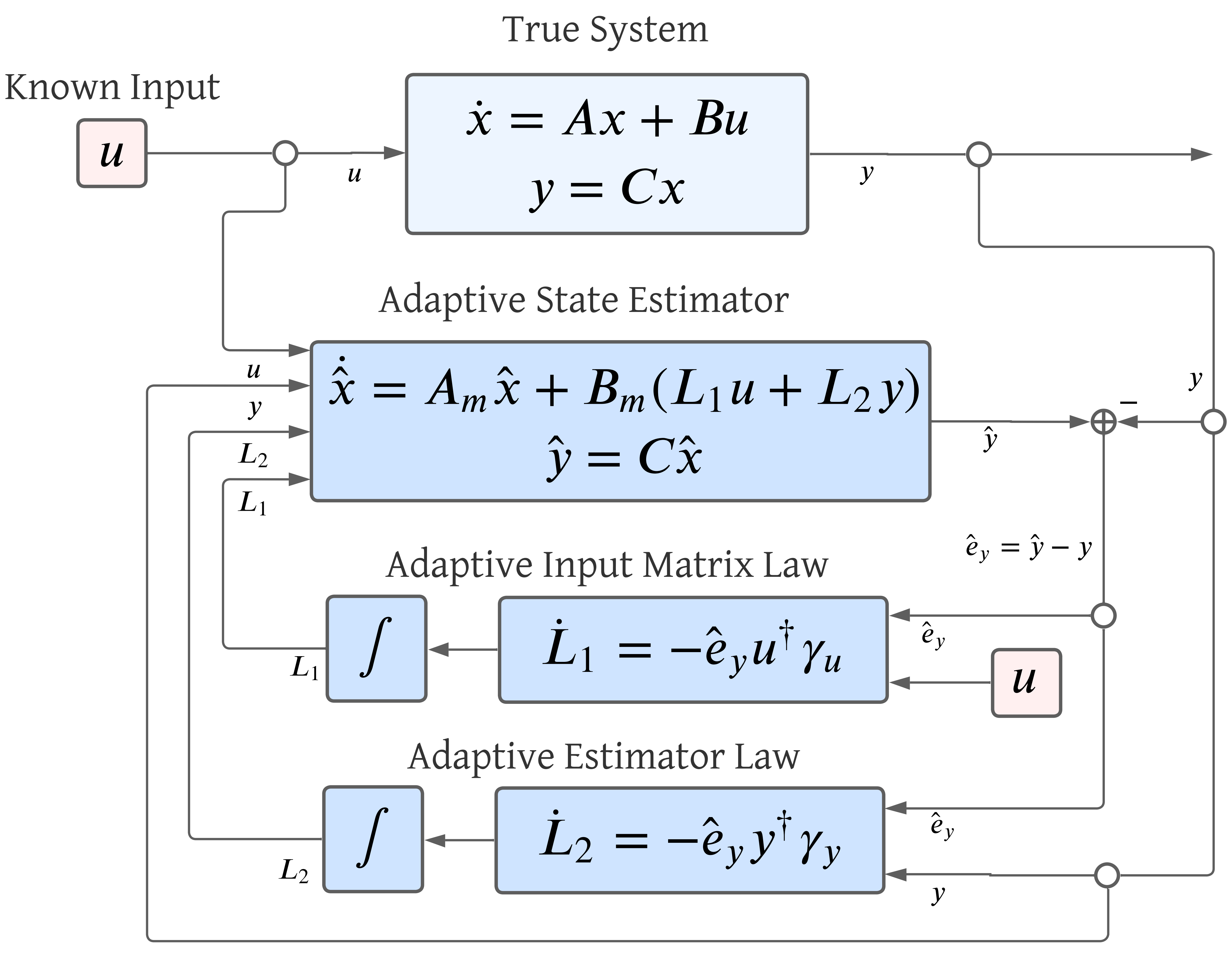


Figure 8: Adaptive State Estimator.

Applying Adaptive State Estimator

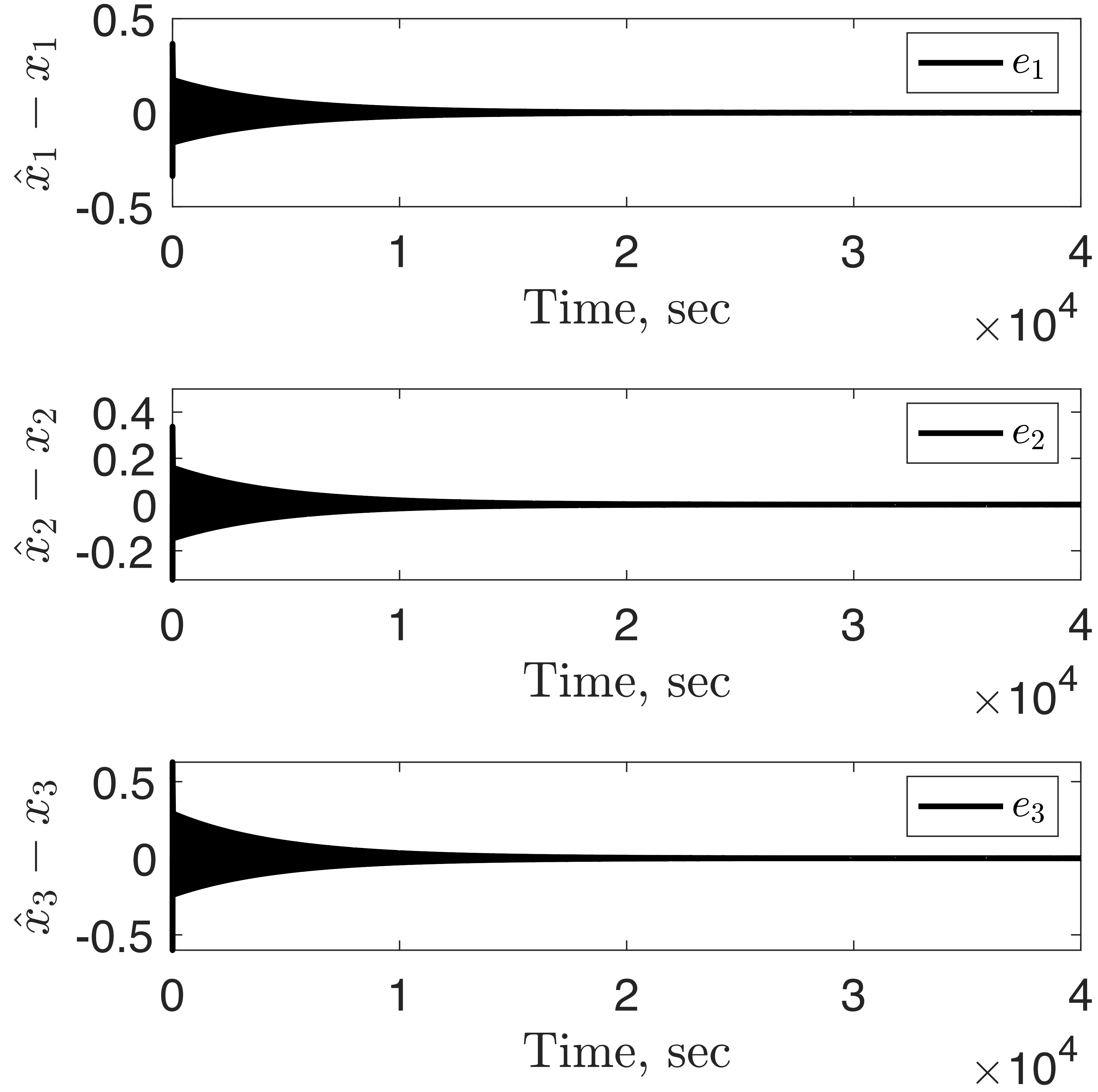


Figure 9: Internal State Error.

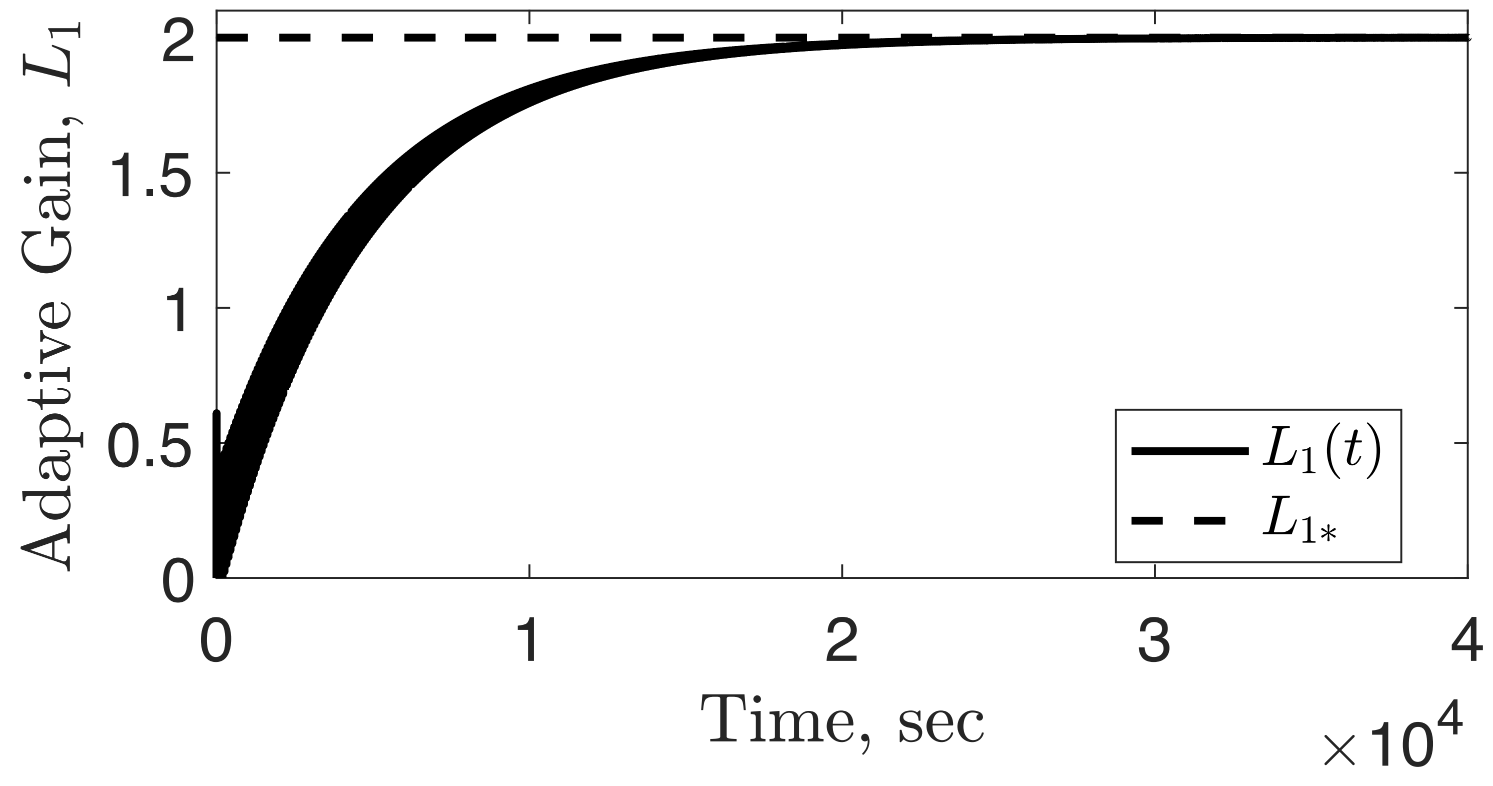


Figure 10: Adaptive Input Matrix Gain Numerically Converging $(L_1(t) \underset{t\to\infty}{\longrightarrow} L_{1*}) \ni B_m L_1(t) C \underset{t\to\infty}{\longrightarrow} B_m L_{1*} = B.$

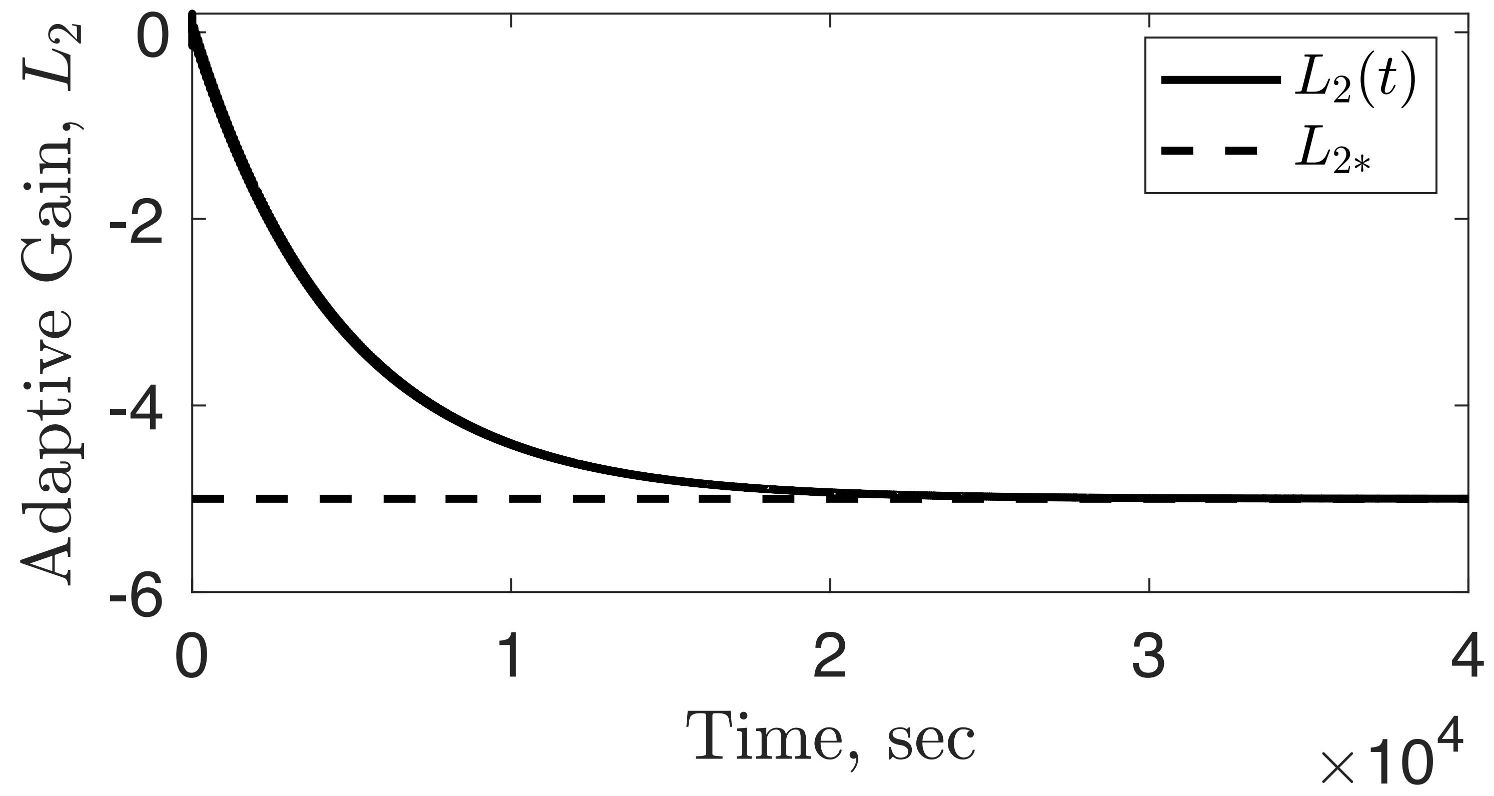


Figure 11: Adaptive Gain Numerically Converging $(L_2(t) \underset{t\to\infty}{\longrightarrow} L_{2*})$ $\ni A_m + BL_2(t)C \underset{t\to\infty}{\longrightarrow} A_m + BL_{2*}C = A.$

Applying Adaptive State Estimator

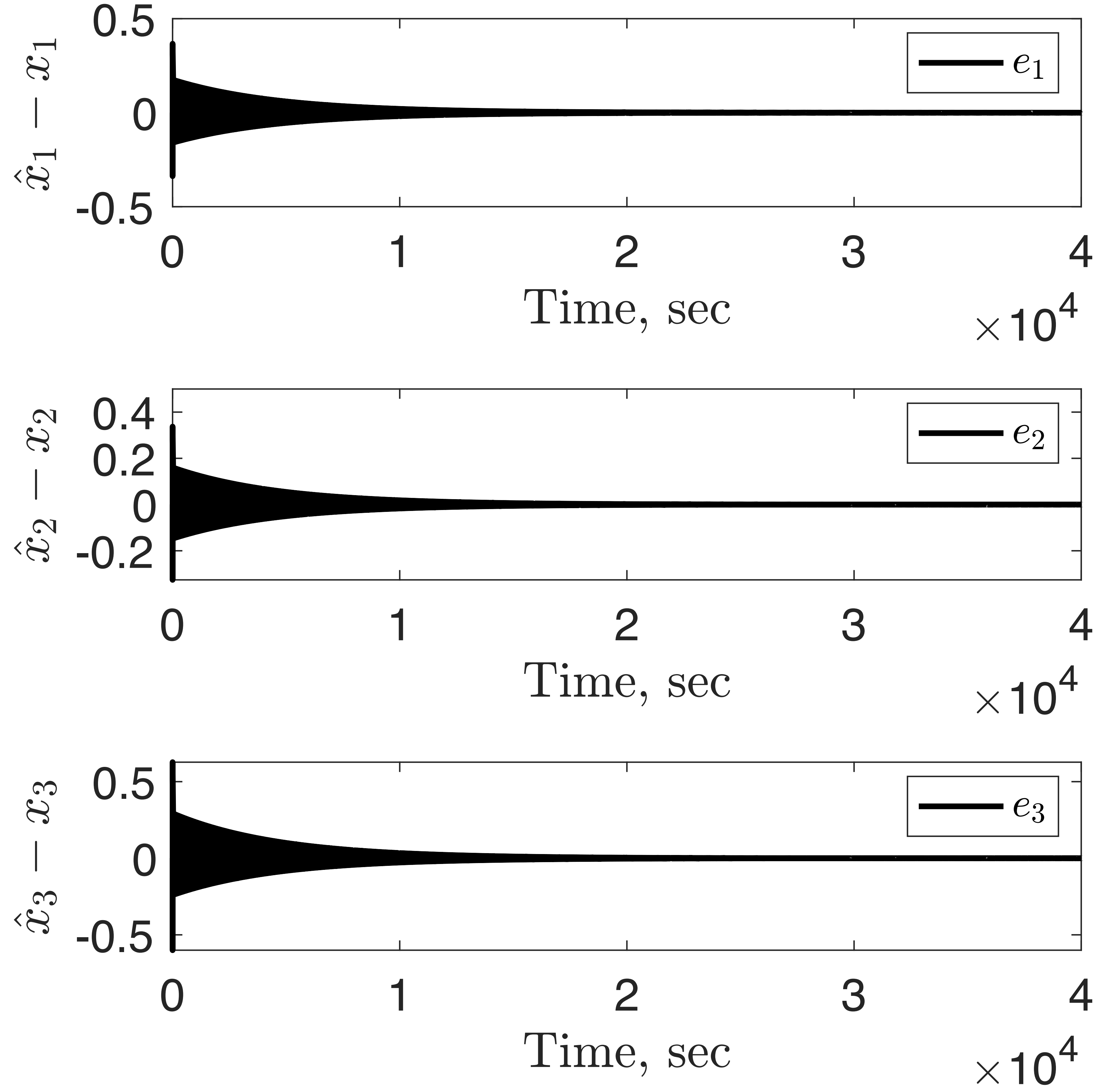


Figure 9: Internal State Error.

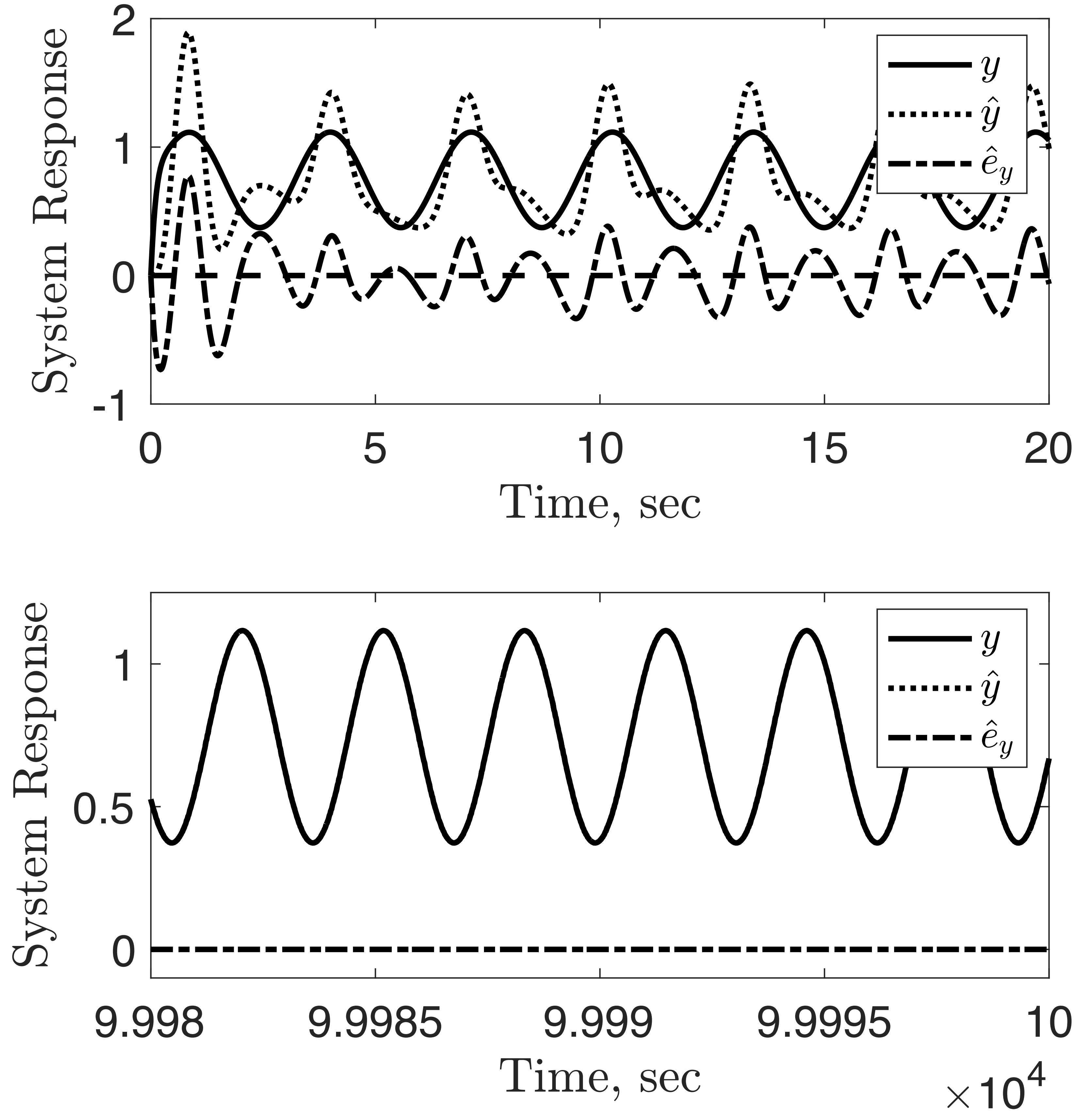


Figure 12: External State Response.

Conclusion

- Given $\{A_m, B_m, C\}$ are known and the true input matrix and plant dynamics follows:

$$B \in \text{sp}\{B_m L_{1*}\} \ni B = B_m L_{1*}$$
$$A \in \text{sp}\{A_m, B_m L_{2*} C\} \ni A = A_m + B_m L_{2*} C$$

- Stability proof guarantees:
  - $e_x \xrightarrow[t\to\infty]{} 0$ and $\hat{e}_y \xrightarrow[t\to\infty]{} 0$ asymptotically.
  - $\{\Delta L_1, \Delta L_2\}$ is guaranteed to bounded.
    - If $\{\Delta L_1, \Delta L_2\} \xrightarrow[t\to\infty]{} 0$ **numerically**, the dynamics of the true input matrix and plant or energy equivalence has be been captured.

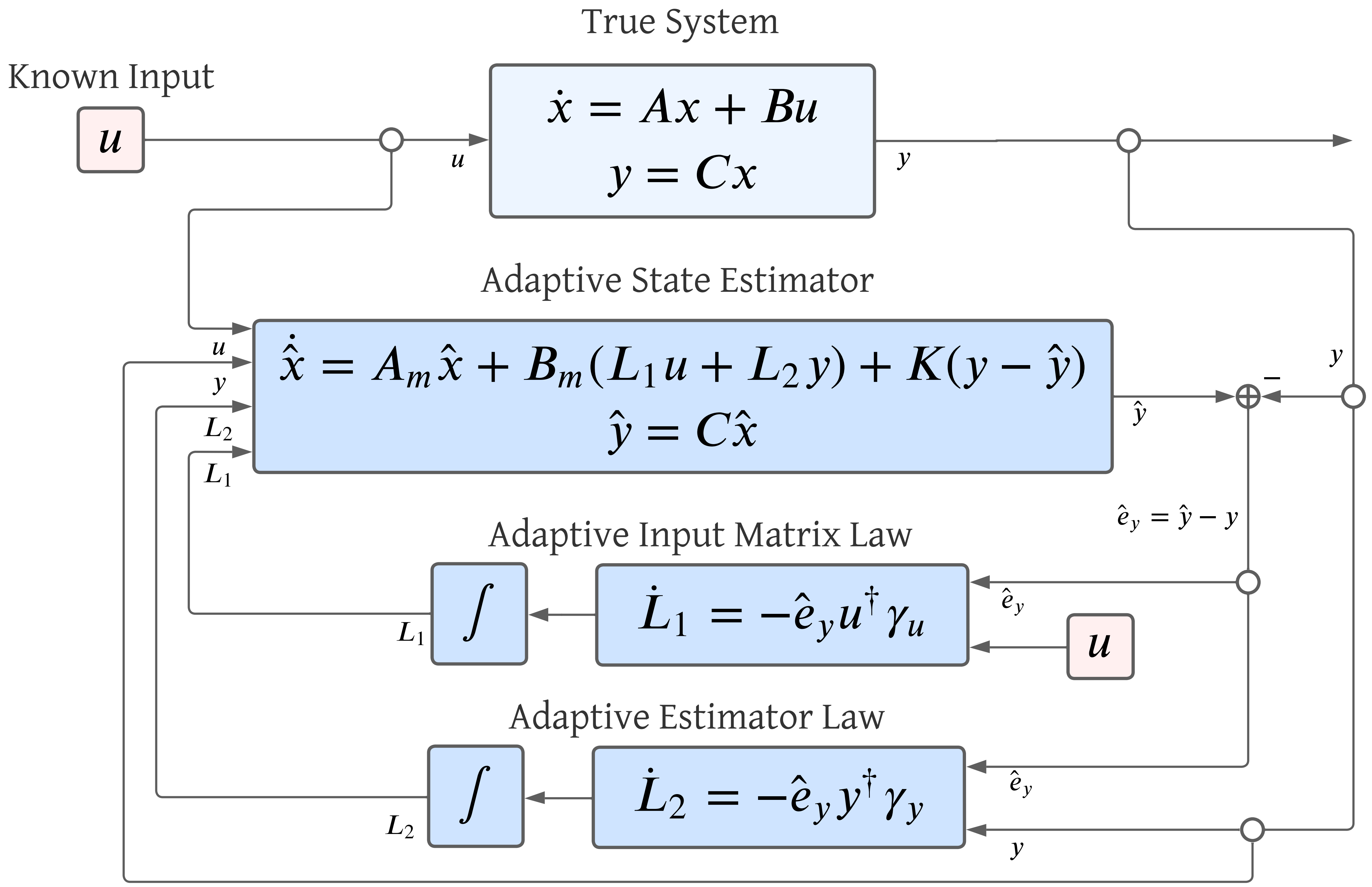


Figure 13: Adaptive State and Input Matrix Estimator.

Thank you!

References

1. Luenberger, David. "An introduction to observers." *IEEE Transactions on automatic control* 16.6 (1971): 596-602.
2. Kalman, Rudolph Emil. "A new approach to linear filtering and prediction problems." (1960): 35-45.
3. Nagpal, Krishan M., and Pramod P. Khargonekar. "Filtering and smoothing in an H/sup infinity/setting." *IEEE Transactions on Automatic Control* 36.2 (1991): 152-166.
4. Doyle, John C. "A review of μ for case studies in robust control." *IFAC Proceedings Volumes* 20.5 (1987): 365-372.
5. Griffith, Tristan, et al. A Modal Approach to the Space-Time Dynamics of Cognitive Biomarkers. 01 2023, https://doi.org/10.1007/978-3-031-23529-0.
6. Fuentes K., Balas M., Hubbard J., 2025, “A Control Framework for Direct Adaptive State and Input Matrix Estimation with Known Inputs”, IARIA.

Appendix

H_∞ Synthesis

- H_∞ Synthesis is a robust controller that uses optimization techniques to determine gains.
  - In practice, control gain are calculated based on the selected input signals the controller has access to.
  - Controller will be optimal relative to the cost function and prescribed input signals. Need not mean controller is optimal for the entire system.
- Depending on the amount of model uncertainty, H_∞ Synthesis could produce a unstable response.

μ Synthesis

- μ Synthesis is an extension of H_∞ Synthesis.
    - The main difference, μ Synthesis account for model uncertainty.
    - In practice, H_∞ Synthesis is ran iteratively until nominal controller is found.
    - Then, the robustness of the controller is tested and assigned a score.
    - Depending on model uncertainty, cycle is repeated until robustness score is minimized.

Defining Error

- To determine the difference between the model and true system, consider the following state and output error equations

$$\begin{cases} e_x = \hat{x} - x \\ \hat{e}_y = Ce_x = C(\hat{x} - x) = \hat{y} - y \end{cases}$$

- Take the time derivative of e_x and plug in error dynamics to determine error convergence

$$\begin{aligned} \dot{e}_x = \dot{\hat{x}} - \dot{x} &= A_m\hat{x} + B(u + Ly) - (Ax + Bu) \\ &= A_m\hat{x} + B(\Delta L + L_*)y - \underbrace{(A_m + BL_*C)}_{A}x \\ &= A_m e_x + B \underbrace{\Delta L y}_{w_x} \end{aligned}$$

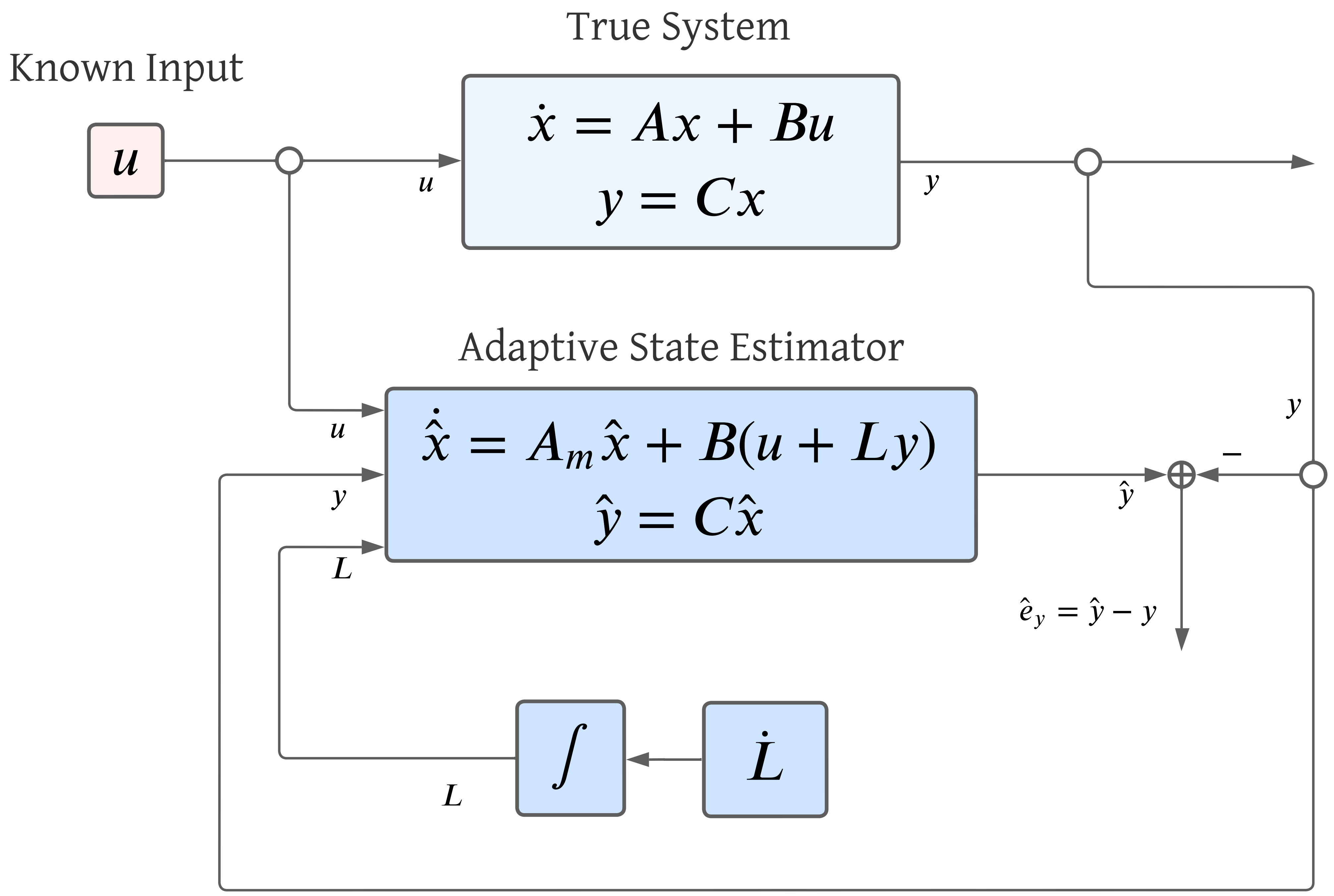
Error Dynamics

- Therefore, the state error dynamics can be written as

$$\begin{cases} \dot{e}_x = A_m e_x + B w_x \\ \hat{e}_y = C e_x \end{cases}$$

- No guarantee that $e_x \underset{t \to \infty}{\longrightarrow} 0$ because of the residual term (Bw_x) in the error equation.

- An additional argument is needed to remove the residual term (Bw_x)



Lyapunov Stability

- Why do we care?
  - Lyapunov argument considered dynamic system's in terms of energy-like functions
  - In this case, we are considering the energy rate of change for the error state to guarantee $e_x \xrightarrow[t \to \infty]{} 0$
    - If error energy is removed, estimator converges to the true plant and state.

Lyapunov Function for the Error System

- Given the following error system

$$\begin{cases} \dot{e}_x = A_m e_x + B w_x \\ \hat{e}_y = C e_x \end{cases}$$

- Assuming real scalars, consider the following Lyapunov function

$$V_e(e_x) = \frac{1}{2} e_x^\dagger P_x e_x ; P_x > 0$$

- Where $V_e(e_x)$ acts as the energy-like function for the error system.

Lyapunov Error Dynamics

- To determine the energy-like rate of change, take the time derivative of $V_e(e_x) = \frac{1}{2}e_x^\dagger P_x e_x$ and plugging in error dynamics

$$\begin{aligned}
2\dot{V}_e &= \dot{e}_x^\dagger P_x e_x + e_x^\dagger P_x \dot{e}_x \\
&= (A_m e_x + B w_x)^\dagger P_x e_x + e_x^\dagger P_x (A_m e_x + B w_x) \\
&= e_x^\dagger (A_m^\dagger P_x + P_x A_m) e_x + 2 e_x^\dagger P_x B w_x
\end{aligned}$$

Lyapunov Error Dynamics cont.

- From the SPR condition,

$$\text{SPR Condition} \begin{cases} A_m^\dagger P_x + P_x A_m < -Q_x \\ P_x B = C^\dagger \end{cases}$$

- $\dot{V}_e(e_x)$ becomes

$$\begin{aligned} 2\dot{V}_e &= e_x^\dagger \underbrace{(A_m^\dagger P_x + P_x A_m)}_{-Q_x} e_x + 2e_x^\dagger \underbrace{P_x B}_{C^\dagger} w_x \\ &= -e_x^\dagger Q_x e_x + 2 \underbrace{e_x^\dagger C^\dagger}_{\hat{e}_y^\dagger} w_x \\ &= -e_x^\dagger Q_x e_x + 2 \underbrace{(\hat{e}_y, w_x)}_{(w_x, \hat{e}_y)} \end{aligned}$$

Lyapunov Error Dynamics Cont.

- The resulting energy-like rate of change Lyapunov Function for the error system becomes

$$\dot{V}_e = -\frac{1}{2}e_x^* Q_x e_x + (\hat{e}_y, w_x); Q > 0$$

- Removing the residual $(\hat{e}_y, w_x)$ term in the above equation will cause $\dot{V}_e \leq 0$

Creating an Additional "Outlet"

- To remove the residual $(\hat{e}_y, w_x)$ term, consider another energy-like function

$$V_L(\Delta L) = \frac{1}{2}\text{tr}(\Delta L \gamma_y^{-1} \Delta L^\dagger); \gamma_y > 0$$

- To determine energy-like rate of change, take the time derivative of $V_L(\Delta L)$

$$\dot{V}_L(\Delta L) = \text{tr}(\Delta \dot{L} \gamma_y^{-1} \Delta L^\dagger)$$

Creating an Additional "Outlet" cont.

- Lets define $\Delta\dot{L} = -\,\hat{e}_y y^\dagger \gamma_y$ and plug into $\dot{V}_L(\Delta L)$

$$\dot{V}_L(\Delta L) = \text{tr}(\underbrace{-\hat{e}_y y^\dagger \gamma_y}_{\Delta\dot{L}} \gamma_y^{-1} \Delta L^\dagger)$$

$$= \text{tr}(-\hat{e}_y \underbrace{y^\dagger \Delta L^\dagger}_{w_x^\dagger})$$

$$= -\text{tr}(w_x^\dagger \hat{e}_y) = -\,w_x^\dagger \hat{e}_y$$

$$= -\,(w_x, \hat{e}_y) = -\,(\hat{e}_y, w_x)$$

Combining Lyapunov Functions

- The closed loop energy-like Lyapunov Functions function can be written as

$$V_{eL} = V_e(e_x) + V_L(\Delta L) = \frac{1}{2}e_x^\dagger P_x e_x + \frac{1}{2}\text{tr}(\Delta L \gamma_y^{-1} \Delta L^\dagger)$$

- Closed loop time energy-like time derivative Lyapunov Function can be written as

$$\begin{aligned}\dot{V}_{eL} = \dot{V}_e(e_x) + \dot{V}_L(\Delta L) &= -\frac{1}{2}e_x^\dagger Q_x e_x + (\hat{e}_y, w_x) - (\hat{e}_y, w_x) \\ &= -\frac{1}{2}e_x^\dagger Q_x e_x \leq 0\end{aligned}$$

- $\therefore \dot{V}_{eL}(e_x, \Delta L) \leq 0 \Rightarrow \{e_x, \Delta L\}$ are bounded, but does not guarantee $e_x \underset{t \to \infty}{\longrightarrow} 0$ because of the negative-semi-definite nature of $\dot{V}_{eL}$.

Barbalat-Lyapunov

Barbalat-Lyapunov

- Given these three condition

    1. V is lower bounded

    2. $\dot{V}$ is negative semi-definite

    3. $\dot{V}$ is uniformly continuous in time

    Then $\dot{V} \xrightarrow[t \to \infty]{} 0$.

- The first two conditions are satisfied from the previous derivation.

Uniformly Continuous

- Recall $\dot{V}_{eL} = -\frac{1}{2}e_x^{\dagger}Q_x e_x \leq 0$, now consider $W_{eL} \ni W_{eL} \subseteq \dot{V}_{eL}$

$$W_{eL} = e_x^{\dagger}Q_x e_x$$

- Taking the time derivative of W_{eL} and plugging in the error dynamics

$$\begin{aligned} \dot{W}_{eL} &= e_x^{\dagger}Q_x \dot{e}_x \\ &= e_x^{\dagger}Q_x(A_m e_x + B\Delta L y) \end{aligned}$$

- From previous result, $\{e_x, \Delta L\}$ is bounded.
- For $\dot{W}_{eL}$ to be bounded, the output (y) must be bounded.
    - Output response will be bounded for any stable plant by showing global exponential stability for the internal states
- By definition, if $\dot{W}_{eL}$ is bounded, then W_{eL} is uniformly continuous.

Satisfying Barbalot-Lyapunov

- Ensures $\dot{V}_{eL} \xrightarrow[t \to \infty]{} 0$
- Guarantees $e_x \xrightarrow[t \to \infty]{} 0$ and $\hat{e}_y \xrightarrow[t \to \infty]{} 0$ asymptotically.
- Does not guarantee $\Delta L \xrightarrow[t \to \infty]{} 0$
    - However, if $\Delta L \xrightarrow[t \to \infty]{} 0$, the dynamics of the true plant have been captured or some minimal error equivalence.