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Design of Third-Order Tensorial RLS Adaptive Filtering Algorithms

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Outline

- Introduction
- Nearest Kronecker Product (NKP) Decomposition
- Third-Order Tensor (TOT) Decomposition
- Recursive Least-Squares (RLS) Based on TOT
- Simulation Results
- Conclusions

Introduction

- RLS adaptive filter ⟺ system identification
  - ↳ solution of a linear system of equations
- Conventional RLS algorithm ➔ high complexity
- Challenge → identification of long length impulse responses (e.g., network/acoustic echo paths)
  - large matrix ➔ complexity / numerical issues
- Decomposition-based approach ⟺ smaller matrices
  - ↳ faster convergence, lower complexity
- *In this paper*: **RLS adaptive filter** using nearest Kronecker product (NKP) and **third-order tensor decomposition (TOT)** of the impulse response

Nearest Kronecker Product (NKP) Decomposition

! Solution for long-length filters:

→ decomposition of impulse responses (⊗ ➔ Kronecker product)

- Impulse response **h** of length $L = L_1 L_2$
- Low-rank systems (e.g., echo paths):

$$\overset{L}{\mathbf{h}} = \sum_{p=1}^{P} \overset{L_2}{\mathbf{h}_{2,p}} \otimes \overset{L_1}{\mathbf{h}_{1,p}}, \quad \underset{\text{decomposition parameter}}{P} < L_2 \qquad \Rightarrow \qquad \underset{\text{(estimate)}}{\overset{L}{\hat{\mathbf{h}}} = \sum_{p=1}^{P} \overset{L_2}{\hat{\mathbf{h}}_{2,p}} \otimes \overset{L_1}{\hat{\mathbf{h}}_{1,p}}}$$

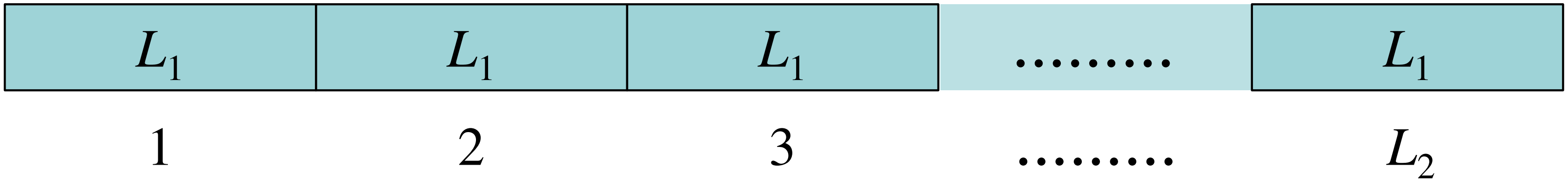
- Decomposition-based approach: $L = L_1 L_2$ ➔ $PL_1 + PL_2$

➔ reformulating a high-dimension system identification problem as a combination of low-dimension solutions.

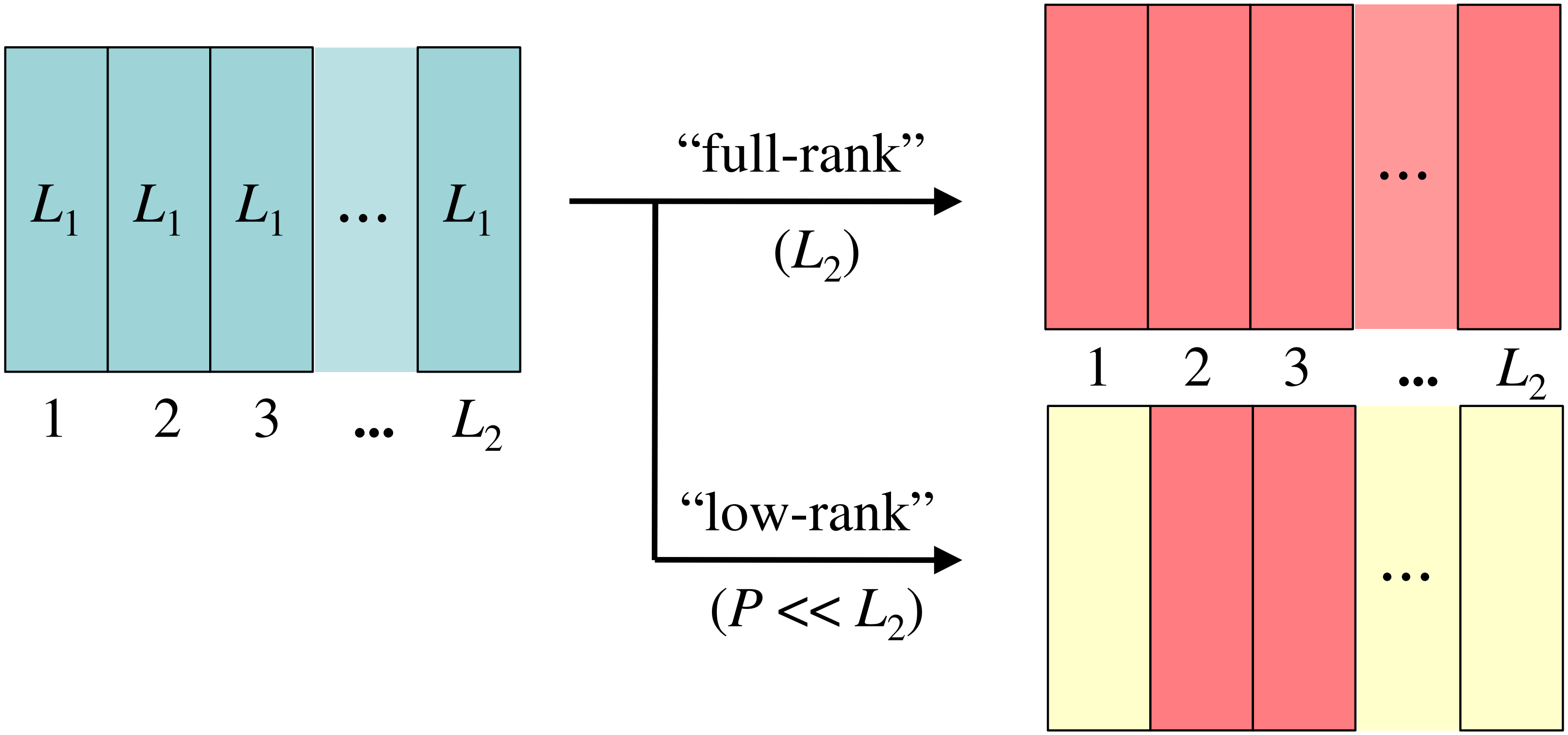
[Paleologu, Benesty, Ciochină, "Linear system identification based on a Kronecker product decomposition," ***IEEE Trans. Audio, Speech, Language Process.***, 2018]

NKP Decomposition (cont.)

- Impulse response **h** of length $L = L_1 L_2$



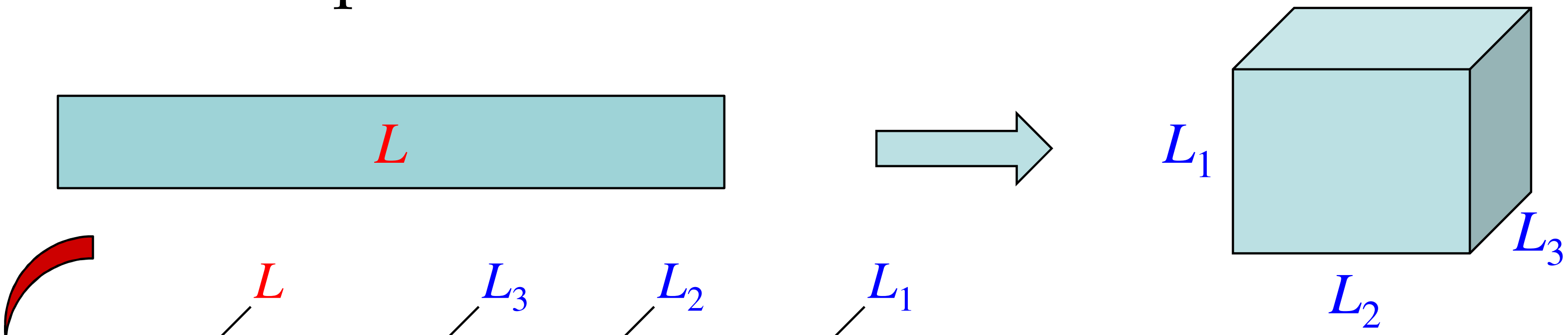
- Reshape vector **h** ➔ **H** - matrix $L_1 \times L_2$



- NKP $\Longleftrightarrow$ singular value decomposition (SVD) of **H**

Third-Order Tensor (TOT) Decomposition

- Impulse response **h** of length $L = L_1 L_2 L_3$
- Reshape vector **h** ➔ $\mathcal{H}$ – third-order tensor L_1 x L_2 x L_3



$$\overset{L}{\mathbf{h}} = \sum_{p=1}^{P} \overset{L_3}{\mathbf{h}_{3,p}} \otimes \overset{L_2}{\mathbf{h}_{2,p}} \otimes \overset{L_1}{\mathbf{h}_{1,p}}, \quad P = ?$$

rank of the tensor

outer product

$$\mathcal{H} = \sum_{p=1}^{P} \mathbf{h}_{1,p} \circ \mathbf{h}_{2,p} \circ \mathbf{h}_{3,p}$$

- **!** Finding the rank of a TOT is a challenging task
- **!** Challenge: avoiding approximation techniques

[Benesty, Paleologu, Ciochină, “Linear system identification based on a third-order tensor decomposition,” ***IEEE Signal Processing Letters***, 2023]

TOT Decomposition (cont.)

- Impulse response **h** of length $L = L_1 L_2$ with $L_1 >> L_2$ and $L_1 = L_{11} L_{12}$

$$\underset{L\nearrow}{\mathbf{h}} = \sum_{i=1}^{L_2} \underset{\nwarrow L_2}{\mathbf{h}_2^i} \otimes \underset{\nwarrow L_1}{\mathbf{h}_1^i} \qquad \underset{L_1\nearrow}{\mathbf{h}_1^i} = \sum_{j=1}^{L_{12}} \underset{\nwarrow L_{12}}{\mathbf{h}_{12}^{ij}} \otimes \underset{\nwarrow L_{11}}{\mathbf{h}_{11}^{ij}} \Longrightarrow \mathbf{h} = \sum_{i=1}^{L_2} \sum_{j=1}^{L_{12}} \mathbf{h}_2^i \otimes \mathbf{h}_{12}^{ij} \otimes \mathbf{h}_{11}^{ij}$$

- Consider that $\mathbf{h}_1^i$ is low-rank $\Longrightarrow$ $\mathbf{h}_1^i = \sum_{p=1}^{P} \mathbf{h}_{12}^{ip} \otimes \mathbf{h}_{11}^{ip}, \quad P < L_{12}$

$$\Longrightarrow \underset{L\nearrow}{\mathbf{h}} = \sum_{l=1}^{L_2} \sum_{p=1}^{P} \underset{\nwarrow L_2}{\mathbf{h}_2^l} \otimes \underset{\nwarrow L_{12}}{\mathbf{h}_{12}^{lp}} \otimes \underset{\nwarrow L_{11}}{\mathbf{h}_{11}^{lp}} \Longleftrightarrow \underset{\boxed{L_{11} \times L_{12} \times L_2}\nearrow}{\mathcal{H}} = \sum_{p=1}^{P} \underbrace{\sum_{l=1}^{L_2} \mathbf{h}_{11}^{lp} \circ \mathbf{h}_{12}^{lp} \circ \mathbf{h}_2^l}_{\text{TOT of rank } L_2}$$

(no approximation) ☑

$$\Longrightarrow \mathcal{H} = \sum_{p=1}^{P} \mathcal{H}_p$$

(sum of P TOTs of rank L_2)

$$\mathbf{h}(L) \quad \Rightarrow \quad \mathbf{h}_2^l\left(L_2^2\right) \,\&\, \mathbf{h}_{12}^{lp}\left(PL_{12}\right) \,\&\, \mathbf{h}_{11}^{lp}\left(PL_{11}\right) \qquad L_2 \ll L_{11}L_{12}, \quad P \ll L_{12}$$

$L = L_{11} L_{12} L_2$ ➔ $L_2^2 + PL_{11} + PL_{12}$ (reduced number of parameters)

RLS Based on TOT

- Goal: "extract" / "separate" the individual components:

$$\mathbf{h}_2^l \quad \& \quad \mathbf{h}_{12}^{lp} \quad \& \quad \mathbf{h}_{11}^{lp} \quad \left(l = 1, \ldots, L_2,\ p = 1, \ldots, P\right)$$

- Extraction of $\mathbf{h}_2^l \quad \left(l = 1, \ldots, L_2\right)$:

$$\mathbf{h} = \sum_{l=1}^{L_2} \sum_{p=1}^{P} \mathbf{h}_2^l \otimes \mathbf{h}_{12}^{lp} \otimes \mathbf{h}_{11}^{lp} = \sum_{l=1}^{L_2} \sum_{p=1}^{P} \left(\mathbf{I}_{L_2} \otimes \mathbf{h}_{12}^{lp} \otimes \mathbf{h}_{11}^{lp}\right) \mathbf{h}_2^l$$

$$= \sum_{l=1}^{L_2} \sum_{p=1}^{P} \mathbf{H}_{12,11}^{lp} \mathbf{h}_2^l = \sum_{l=1}^{L_2} \overline{\mathbf{H}}_{12,11}^{l} \mathbf{h}_2^l = \underline{\underline{\overline{\mathbf{H}}}_{12,11} \underline{\mathbf{h}}_2}$$

$$\mathbf{H}_{12,11}^{lp} = \mathbf{I}_{L_2} \otimes \mathbf{h}_{12}^{lp} \otimes \mathbf{h}_{11}^{lp}, \quad \overline{\mathbf{H}}_{12,11}^{l} = \sum_{p=1}^{P} \mathbf{H}_{12,11}^{lp}$$

$$\underline{\overline{\mathbf{H}}}_{12,11} = \begin{bmatrix} \overline{\mathbf{H}}_{12,11}^{1} & \cdots & \overline{\mathbf{H}}_{12,11}^{L_2} \end{bmatrix}, \quad \underline{\mathbf{h}}_2 = \begin{bmatrix} \left(\mathbf{h}_2^1\right)^T & \cdots & \left(\mathbf{h}_2^{L_2}\right)^T \end{bmatrix}^T$$

RLS Based on TOT (cont.)

- Extraction of $\mathbf{h}_{12}^{lp}$ & $\mathbf{h}_{11}^{lp}$ $\left(l=1,\ldots,L_2,\ p=1,\ldots,P\right)$

$$\mathbf{h}=\sum_{l=1}^{L_2}\sum_{p=1}^{P}\mathbf{h}_2^l\otimes\mathbf{h}_{12}^{lp}\otimes\mathbf{h}_{11}^{lp}=\sum_{l=1}^{L_2}\sum_{p=1}^{P}\left(\mathbf{h}_2^l\otimes\mathbf{I}_{L_{12}}\otimes\mathbf{h}_{11}^{lp}\right)\mathbf{h}_{12}^{lp}=\ldots=\underline{\underline{\overline{\mathbf{H}}}_{2,11}\underline{\overline{\mathbf{h}}}_{12}}$$

$$=\sum_{l=1}^{L_2}\sum_{p=1}^{P}\left(\mathbf{h}_2^l\otimes\mathbf{h}_{12}^{lp}\otimes\mathbf{I}_{L_{11}}\right)\mathbf{h}_{11}^{lp}=\ldots=\underline{\underline{\overline{\mathbf{H}}}_{2,12}\underline{\overline{\mathbf{h}}}_{11}}$$

- Notation: $\mathbf{g}_*$ and $\mathbf{G}_*$ → estimates of $\mathbf{h}_*$ and $\mathbf{H}_*$, respectively

$$e\left(n\right)=d\left(n\right)-\mathbf{g}^T\left(n-1\right)\mathbf{x}\left(n\right) \Longrightarrow \text{Least-squares (LS) criterion}$$

$$=d\left(n\right)-\underline{\mathbf{g}}_2^T\left(n-1\right)\boxed{\underline{\overline{\mathbf{G}}}_{12,11}^T\left(n-1\right)\mathbf{x}\left(n\right)}=d\left(n\right)-\underline{\mathbf{g}}_2^T\left(n-1\right)\boxed{\mathbf{x}_{12,11}\left(n\right)}=e_1\left(n\right)$$

$$=d\left(n\right)-\underline{\overline{\mathbf{g}}}_{12}^T\left(n-1\right)\boxed{\underline{\overline{\mathbf{G}}}_{2,11}^T\left(n-1\right)\mathbf{x}\left(n\right)}=d\left(n\right)-\underline{\overline{\mathbf{g}}}_{12}^T\left(n-1\right)\boxed{\mathbf{x}_{2,11}\left(n\right)}=e_2\left(n\right)$$

$$=d\left(n\right)-\underline{\overline{\mathbf{g}}}_{11}^T\left(n-1\right)\boxed{\underline{\overline{\mathbf{G}}}_{2,12}^T\left(n-1\right)\mathbf{x}\left(n\right)}=d\left(n\right)-\underline{\overline{\mathbf{g}}}_{11}^T\left(n-1\right)\boxed{\mathbf{x}_{2,12}\left(n\right)}=e_3\left(n\right)$$

RLS Based on TOT (cont.)

- LS cost functions ⇨ Normal equations

$$\left\{\begin{array}{lll}
J\left(\underline{\mathbf{g}}_2 \middle| \overline{\underline{\mathbf{g}}}_{12}, \overline{\underline{\mathbf{g}}}_{11}\right) & \Rightarrow \quad \mathbf{R}_{12,11}(n)\,\underline{\mathbf{g}}_2(n) = \mathbf{r}_{12,11}(n) & \Rightarrow \quad \underline{\mathbf{g}}_2(n) \\
J\left(\overline{\underline{\mathbf{g}}}_{12} \middle| \underline{\mathbf{g}}_2, \overline{\underline{\mathbf{g}}}_{11}\right) & \Rightarrow \quad \mathbf{R}_{2,11}(n)\,\overline{\underline{\mathbf{g}}}_{12}(n) = \mathbf{r}_{2,11}(n) & \Rightarrow \quad \overline{\underline{\mathbf{g}}}_{12}(n) \\
J\left(\overline{\underline{\mathbf{g}}}_{11} \middle| \underbrace{\underline{\mathbf{g}}_2, \overline{\underline{\mathbf{g}}}_{12}}_{\text{fixed}}\right) & \Rightarrow \quad \mathbf{R}_{2,12}(n)\,\overline{\underline{\mathbf{g}}}_{11}(n) = \mathbf{r}_{2,12}(n) & \Rightarrow \quad \overline{\underline{\mathbf{g}}}_{11}(n)
\end{array}\right.$$

➔ RLS adaptive algorithm using TOT decomposition **(RLS-TOT)**

- Final estimate:

$$\mathbf{g}(n) = \sum_{l=1}^{L_2}\sum_{p=1}^{P} \mathbf{g}_2^{l}(n) \otimes \mathbf{g}_{12}^{lp}(n) \otimes \mathbf{g}_{11}^{lp}(n)$$

Advantages

- smaller data structures (matrices)
- faster convergence/tracking
- lower computational complexity

Simulation Results

- **Conditions:**

  → **h** from ITU-T Rec. G168, with $L = 512$.

  → TOT decomposition: $L_{11} = L_{12} = 16$, $L_2 = 2$

  → **h** acoustic impulse response, with $L = 2048$.

  → TOT decomposition: $L_{11} = L_{12} = 32$, $L_2 = 2$

  → input signal – AR(1) process with pole at 0.8 / speech sequence

  → additive noise – white Gaussian noise, with SNR = 20 or 10 dB.

  → performance measure: normalized misalignment (dB).

$$20\log_{10}\left[\left\|\mathbf{h}-\mathbf{g}(n)\right\|_2 / \left\|\mathbf{h}\right\|_2\right]$$

- **Algorithms:**

→ conventional **RLS**

→ **RLS-TOT**

→ **RLS-NKP / APA / DR-FRLS** (see the references [6] / [12] / [13])

Simulation Results (cont.)

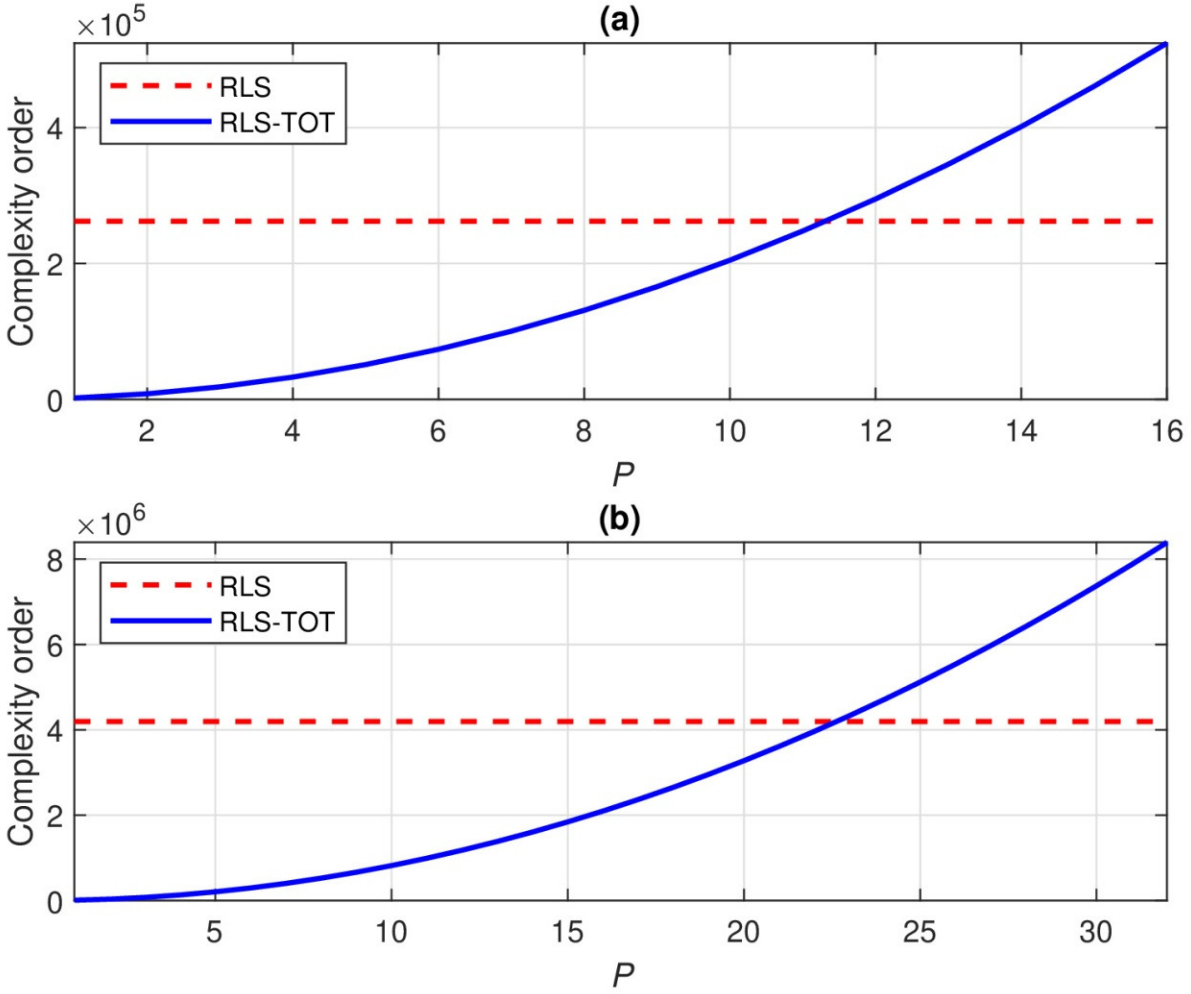


Figure 1. Complexity order of the conventional RLS algorithm and RLS-TOT for two impulse responses, with lengths (a) $L = 512$ and (b) $L = 2048$.

Simulation Results (cont.)

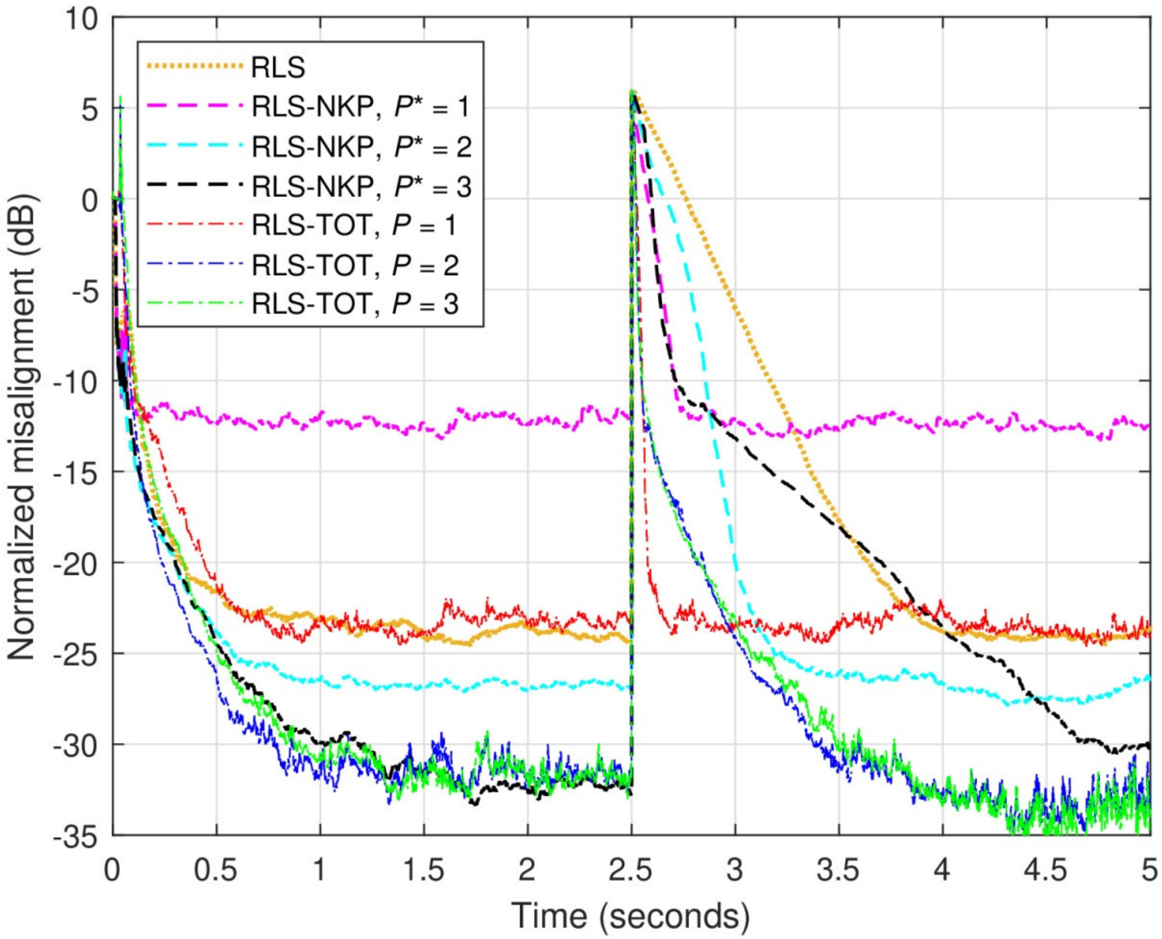


Figure 2. Misalignment of the RLS-based algorithms for the identification of a network impulse response of length $L = 512$. The forgetting factors are set based on equation (4), using $K = 5$ for the conventional RLS algorithm, and $K = 45$ for the RLS-NKP and RLS-TOT. The input signal is an AR(1) process and SNR = 20 dB.

Simulation Results (cont.)

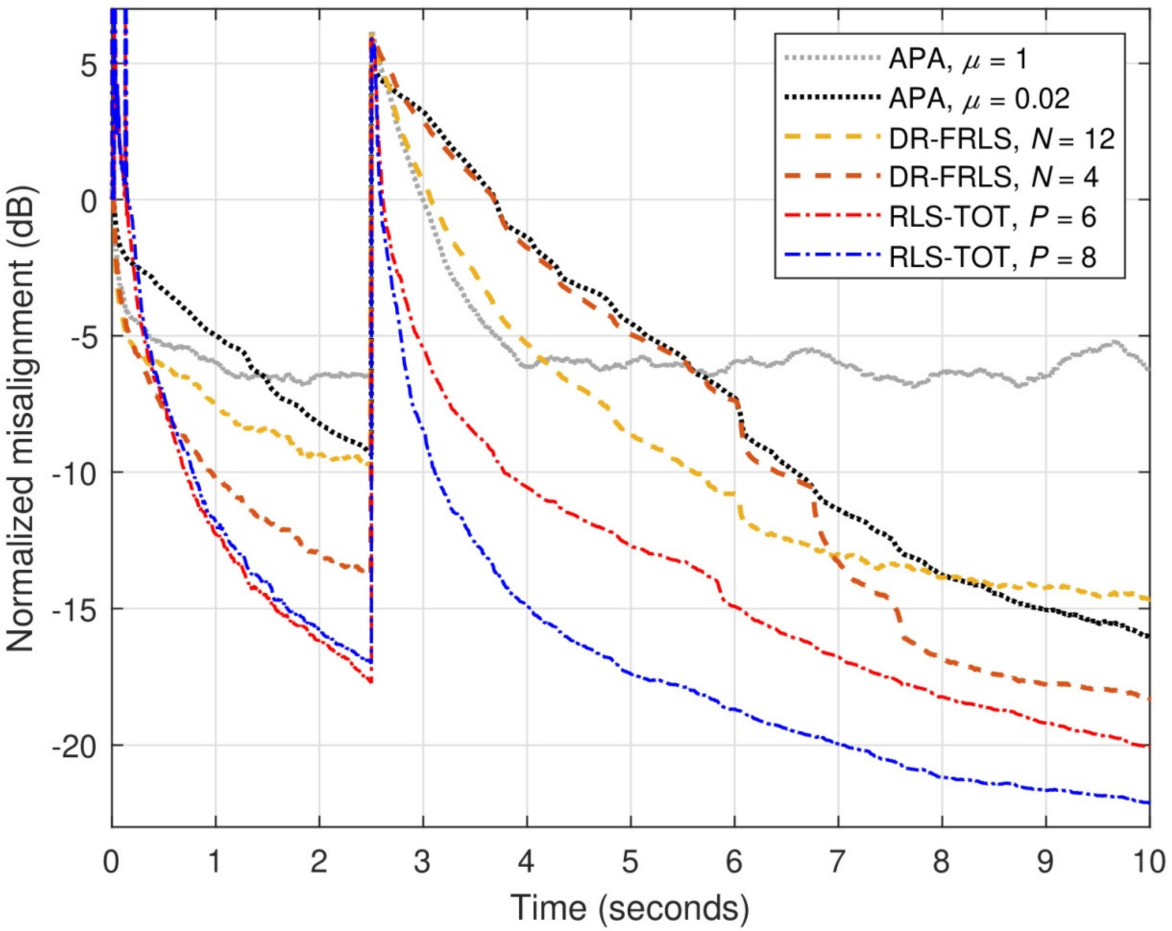


Figure 3. Misalignment of the APA, DR-FRLS algorithm, and RLS-TOT, for the identification of an acoustic impulse response of length $L = 2048$. The RLS-TOT uses two forgetting factors set based on equation (4), with $K = 100$, while the third one is equal to 1. The input signal is speech and SNR = 10 dB.

Simulation Results (cont.)

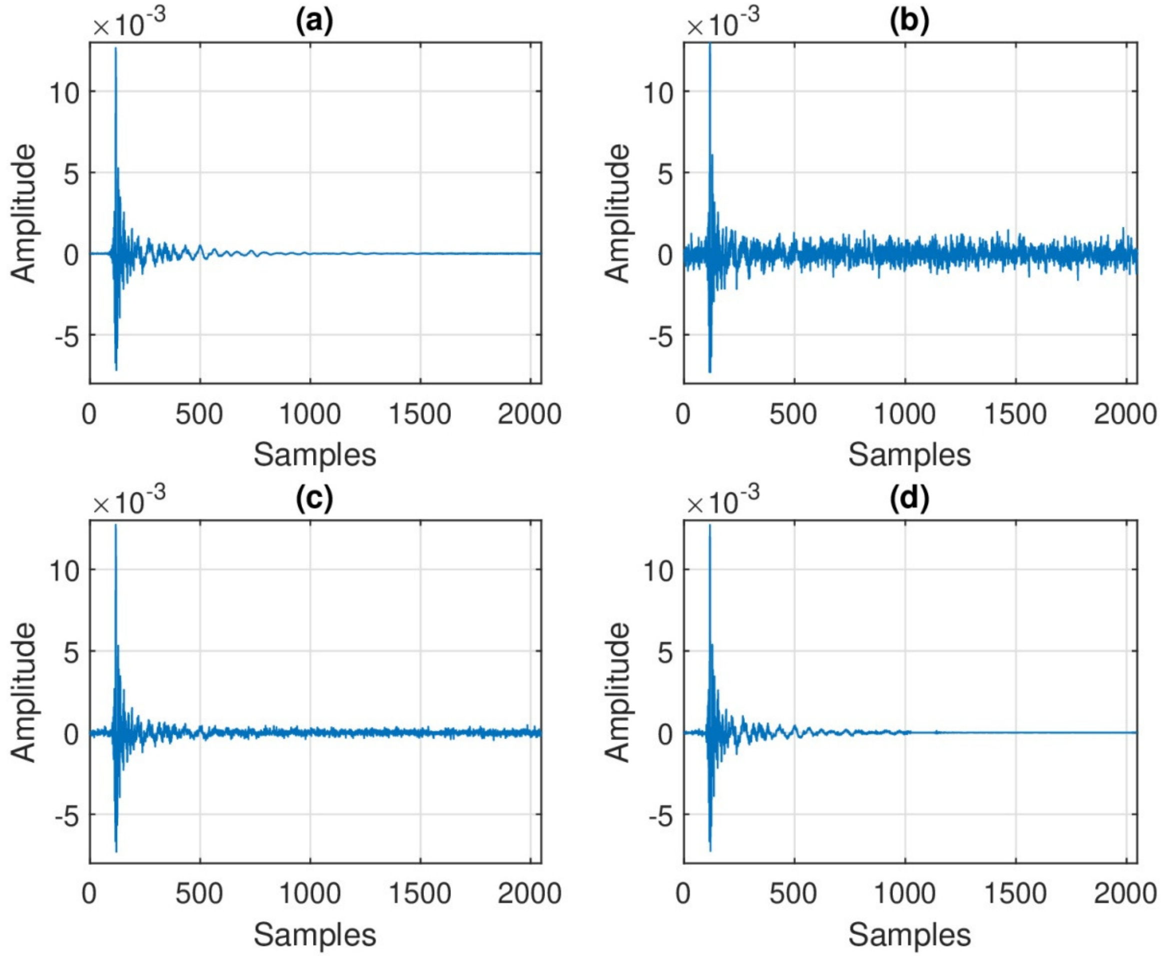


Figure 4. Impulse responses related to the experiment reported in Figure 3: (a) true acoustic impulse response **h**; (b) the estimate obtained by APA using the step-size equal to 1; (c) the estimate obtained by DR-FRLS using the data-reuse parameter equal to 12; and (d) the estimate obtained by RLS-TOT using $P = 8$.

Conclusions and Perspectives

- System identification exploiting a third-order tensor (TOT) decomposition.
- Efficient solution for the identification of long-length low-rank systems (e.g., echo paths).
- High-dimension system identification problem → reformulated as a combination of low-dimension solutions (three shorter filters).
- *Solution*: RLS adaptive filter based on TOT ➔ RLS-TOT.
- The RLS-TOT outperforms the conventional RLS and other RLS-based algorithms (faster convergence/tracking & lower computational complexity).
- **Future works**: - dichotomous CD (DCD) → reduce complexity.

- extension to multidimensional case → higher-order tensors.

- improved versions with variable forgetting factors and variable regularization parameters.

ACKNOWLEDGMENT

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