Mixture based hybrid regularization method for blind image deconvolution

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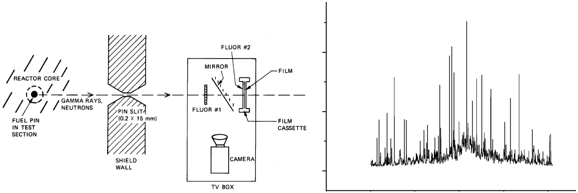
Neutron radiography (NR)



Fg: 1. a) Sketch map of NR system. b) Line slices of a NR data

- NR provides invaluable and complementary information to flash X/γ-ray radiography
- **Blur and noise** in NR/FXR systems are introduced by components of the imaging system, and this in turn produces compositions of distributions for their models
- **Heavy-tailed very impulsive components** must be taken into account for better radiograph modeling

Introduction

Cauchy (or Cauchy-Lorenz) distribution with PDF

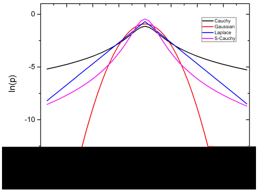
$$p_c(y;\mu,\sigma^2) = \frac{\sigma}{\pi(\sigma^2 + |y-\mu|^2)}, \tag{1}$$

where $\sigma > 0$ and $\mu \in \mathbb{R}$ are known as distance or scale parameter and localization parameter, respectively.

Cauchy distribution:

- widely used to simulate the impulsive behavior appeared in various imaging applications (e.g. SAR, RS);
- utilized to depict the radiation response in FXR;
- If $X, Y \sim N(0,1)$, then $Z = \frac{X}{Y} \sim P_C(0,1)$; If $X \sim P_C(0,1)$, then EX^r does not exist for $r \geq 1$;
- **NOTORIOUS** for the undesirable attributes of possessing an ***undefined mean*** and an ***infinite variance***.

Introduction



Fg: 2.PDFs of standard Gaussian, S-Cauchy, Cauchy,and Laplace distributions (Log domain)

Square Cauchy (S-Cauchy) distribution:

- PDF is

$$p_{sc}(y;\mu,\sigma^2)=\frac{2\sigma^3}{\pi(\sigma^2+|y-\mu|^2)^2} \tag{2}$$

- possesses **the first and second moments**; similar to Cauchy or Laplace distribution, while it has the highest density at the center
- more appropriate to characterize the impulsive outliers with lower frequency or in a sparser way

New wine in a old bottle

Data model:

$$f = \text{Poisson}(k * u + b) + w \doteq z(u, k) + w, \quad (3)$$

$$z \sim P_Z^{U,K}(k * u + b), w \sim P_{SC}(0, \sigma_w^2) \quad (4)$$

with PDFs

$$p_z^{u,k}(z; Bu + b) = \frac{(Bu + b)^z e^{-(Bu+b)}}{z!}, z \geq 0 \quad (5)$$

and

$$p_{sc}(w; 0, \sigma_w^2) = \frac{2\sigma^3}{\pi(\sigma_w^2 + |w|^2)^2}, \quad (6)$$

where f is a noisy blurred image, u is the source image, $Bu \equiv k * u$, k is the convolution kernel, b is a background constant, $\sigma_w^2 > 0$ is the unknown parameter of S-Cauchy density.

Problem to be studied

Our denoising problem is to recover u from a MPsC noisy data f with unknown parameter σ_w^2.

The data model is reduced to

$$f = z(u) + w \tag{7}$$

$$z \sim P_Z^U(u+b), w \sim P_{SC}(\sigma_w^2) \tag{8}$$

with PDFs

$$p_z^u(z; u+b) = \frac{(u+b)^z e^{-(u+b)}}{z!}, z \geq 0 \tag{9}$$

and

$$p_{sc}(w; 0, \sigma_w^2) = \frac{2\sigma^3}{\pi(\sigma_w^2 + |w|^2)^2}. \tag{10}$$

$A = I$

- Proposed Model: MPsC-$TV^{1,\alpha}$

The **amplitude varying Gaussian PDF** is

$$p_{avg}(y; u+b) = \frac{(u+b)^{\beta}}{C} \exp\left(-\frac{|y-(u+b)|^2}{2(u+b)}\right), \qquad (11)$$

where $\beta > 0$ is a parameter, $C > 0$ is the normalization constant.

Utilizing the MAP procedure, we get the following **optimization model**

$$\min_{u,w,\sigma_w^2} \mathcal{E}(u,w,\sigma_w^2) = \Psi(u,w) + \Phi(w,\sigma_w^2) + R(\nabla u, \nabla^{\alpha} u) \qquad (12)$$

where

$$\Psi(u,w) = \int_{\Omega} \left(\frac{|u+b+w-f|^2}{2(u+b)} - \beta \log(u+b)\right) \mathrm{dx}, \qquad (13)$$

$$\Phi(w,\sigma^2) = \int_{\Omega} (2\log(\sigma^2 + |w|^2) - \frac{3}{2}\log(\sigma_w^2))\mathrm{dx}, \qquad (14)$$

$$R(\nabla u, \nabla^{\alpha} u) = \int_{\Omega} g_1 |\nabla u| \mathrm{dx} + \int_{\Omega} g_2 |\nabla^{\alpha} u|, g_i(\mathrm{x}) > 0, i = 1, 2, \qquad (15)$$

$\alpha \in (1, 2]$.

- Four directional fractional-order GL gradient $\nabla^\alpha u$

The discrete fractional-order gradient transform of u is defined as

$$\nabla^\alpha u = (D_x^\alpha u, D_y^\alpha u, D_d^\alpha u, D_b^\alpha u)^{\mathrm{T}} \doteq (D_1^\alpha u, D_2^\alpha u, D_3^\alpha u, D_4^\alpha u)^{\mathrm{T}}, \tag{16}$$

where $D_i^\alpha u$, $i = 1, 2, 3, 4$ represents fractional-order along horizontal, vertical, diagonal, and back diagonal direction approximated by

$$\begin{cases} D_x^\alpha u(i,j) = \sum_{k=0}^{K-1} (-1)^k C_k^\alpha u(i-k,j) \\ D_y^\alpha u(i,j) = \sum_{k=0}^{K-1} (-1)^k C_k^\alpha u(i,j-k) \end{cases} \tag{17}$$

$$\begin{cases} D_d^\alpha u(i,j) = 2^{-\frac{\alpha}{2}} \sum_{k=0}^{K-1} (-1)^k C_k^\alpha u(i-k,j-k) \\ D_b^\alpha u(i,j) = 2^{-\frac{\alpha}{2}} \sum_{k=0}^{K-1} (-1)^k C_k^\alpha u(i-k,j+k), \end{cases} \tag{18}$$

Here K refers to the number of signals involved in the computation of the fractional-order derivative, and the coefficients $\{C_k^\alpha\}_{k=0}^{K-1}$ are given by

$$C_k^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1-k)}$$

with the Gamma function $\Gamma(x)$.

MPS-TV$^{1,\alpha}$ model

The discrete four directional fractional-order (FOTV4) of u is defined as

$$\|\nabla^\alpha u\|_1 := \sum_{i,j}\sqrt{\sum_{i=1}^{4}|D_k^\alpha u_{i,j}|^2}. \tag{19}$$

According to the relation that $(\nabla^\alpha)^* = (-\bar{1})^\alpha \mathrm{div}^\alpha$, the discrete four directional fractional-order divergence $\mathrm{div}^\alpha \mathrm{p}$ for $\mathrm{p} = (p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)})$ is formulated by

$$(\mathrm{div}^\alpha \mathrm{p})_{i,j} = (-1)^\alpha \sum_{k=0}^{K-1} (-1)^k C_k^\alpha \left(p^{(1)}_{i+k,j} + p^{(2)}_{i,j+k} + 2^{-\frac{\alpha}{2}} (p^{(3)}_{i+k,j+k} + p^{(4)}_{i+k,j-k})\right). \tag{20}$$

In the discrete setting, The regularization term is

$$\sum_{i,j} g_1 \left(\sum_{k=1}^{2} (D_k u_{i,j})^2\right)^{1/2} + \sum_{i,j} g_2 \left(\sum_{k=1}^{2} (D_k^\alpha u_{i,j})^2\right)^{1/2}, \tag{21}$$

where $(D_1 u, D_2 u) = (D_x u, D_y u)$ is defined by common forward difference operators.

Convexity

Lemma

Let Ω *be an open bounded subset of* $\mathbb{R}^2$ *with* $|\Omega| = \int_\Omega 1\mathrm{dx}$. *Assume that* $\sigma^2 > 0,\ u \geq 0,\ b > 0,\ w \in L^4(\Omega)$. $u \in BV_g^\alpha(\Omega)$ *with* $u \geq 0$. g_i *is a continuous function on* Ω, *and is bounded below from zero. Then the functional* $\mathcal{E}(u, w, \sigma^2)$ *is strictly convex W.R.T.* u. *Moreover, if there holds*

$$\frac{u+b}{2} \leq \sigma^2 \leq \frac{3+2\sqrt{3}}{|\Omega|} \int_\Omega |w|^4 \mathrm{dx}, \tag{22}$$

then the functional $\mathcal{E}(u, w, \Theta)$ *is convex W.R.T. the variables* σ^2 *and* w.

$$BV_g^\alpha(\Omega) = \{u \in L^1(\Omega) | TV_g^\alpha(u) < \infty, g(x) > 0, x \in \Omega\} \tag{23}$$

$$TV_g^\alpha(u) := \int_\Omega g|\nabla^\alpha u| = \sup_{\phi \in H_g} \int_\Omega (-u\mathrm{div}^\alpha \phi) dx, \tag{24}$$

$$H_g := \{\phi \in \mathfrak{L}_0^\ell(\Omega, \mathbb{R}^d) : |\phi| \leq g \text{for all} x \in \Omega\}. \tag{25}$$

MPS-TV$^{1,\alpha}$ model

- Numerical framework: BCD-ADMM based algorithm

Following the convexity of our proposed model, we can solve it numerically by **block coefficient descent (BCD)** method of the Gauss-Seidel type. More specifically, it can be solved by this iterative algorithm:

Choose initial guesses for u^0, w^0 and $(\sigma^2)^0$. For $v = 0, 1, 2, \cdots$, do

$$\begin{cases} u^{v+1} & = \arg\min_u \mathcal{E}(u, w^v, (\sigma_w^2)^v), \\ w^{v+1} & = \arg\min_w \mathcal{E}(u^{v+1}, w, (\sigma_w^2)^v), \\ (\sigma_w^2)^{v+1} & = \arg\min_{\sigma_w^2} \mathcal{E}(u^{v+1}, w^{v+1}, \sigma_w^2). \end{cases} \tag{26}$$

Check the convergence, if converged, stop; else goto the first subproblem.

Moreover, a **variable splitting** and the **alternating direction method of multipliers (or ADMM)** are combined with the **Cardano formula (CF)** and anisotropic diffusion to gain computation efficiency and detail preservation.

BCD-ADMM algorithm: Updating of w

We turn to consider the minimization problem

$$\min_{w}\left\{\int_{\Omega}\frac{|w-(f-u^{v+1}-b)|^2}{2(u^{v+1}+b)}\mathrm{dx}+2\int_{\Omega}\log\left((\sigma^2)^v+|w|^2\right)\mathrm{dx}\right\}. \tag{27}$$

For concision, we omit the superscripts and reformulate the integrand to a function $q:\mathbb{R}\to\mathbb{R}$:

$$q(w)=2\log(w^2+\sigma^2)+\frac{|w+u+b-f|^2}{2(u+b)}. \tag{28}$$

The function q is strictly convex for $2\sigma^2/(u+b)>1$, which implies that there has a unique solution to solve the minimization problem. We then consider the solvability of its optimality condition $q'(w)=0$, that is,

$$\frac{4w}{w^2+\sigma^2}+\frac{w+u+b-f}{u+b}=0, \tag{29}$$

or equivalently, a **cubic equation** as follows:

$$w^3+B_v w^2+C_v w+D_v=0, \tag{30}$$

where $B_v=u+b-f$, $C_v=4(u+b)+\sigma^2$, $D_v=B_v\sigma^2$.

BCD-ADMM algorithm: Updating of w

The cubic equation can be solved explicitly by the Cardano formula (CF). Substituting $g - \frac{B_v}{3}$ for w in (30), we have that

$$g^3 + \tilde{p}g + \tilde{q} = 0, \tag{31}$$

where

$$\tilde{p} = C_v - \frac{B_v^2}{3}, \tilde{q} = D_v - \frac{B_v C_v}{3} + \frac{2B_v^3}{27}. \tag{32}$$

Utilizing the CF approach, we can derive the roots of Eq. (31) as follows:

$$\begin{aligned} g_1 &= \sqrt[3]{-\frac{\tilde{q}}{2} + \diamondsuit} + \sqrt[3]{-\frac{\tilde{q}}{2} - \diamondsuit} \doteq \xi + \zeta, \\ g_2 &= \omega\xi + \omega^2\zeta,\ \ g_3 = \omega^2\xi + \omega\zeta, \end{aligned} \tag{33}$$

where $\omega = \exp(\frac{2\pi \mathbf{i}}{3})$, $\mathbf{i}^2 = -1$, and

$$\diamondsuit \doteq \sqrt{(\frac{\tilde{p}}{3})^3 + (\frac{\tilde{q}}{2})^2}. \tag{34}$$

And thus, if $\diamondsuit > 0$, the unique real valued root of Eq. (30) is $\bar{g}_1 = g_1 - \frac{B_v}{3}$, and the minimizer of the subproblem is given by

$$w^{v+1} = \xi + \zeta - \frac{B_v}{3}. \tag{35}$$

BCD-ADMM algorithm: Updating of σ_w^2

Given w^{v+1}. To calculate $(\sigma_w^2)^{v+1}$, we consider $\frac{\partial\Phi}{\partial\sigma_w^2}=0$, which implies that

$$\int_\Omega \frac{\sigma_w^2-3|w^{v+1}|^2}{\sigma_w^2(|w^{v+1}|^2+\sigma_w^2)}\mathrm{dx}=0. \tag{36}$$

According to Proposition, there exists an iteration solving (36). We reuse $(\sigma_w^2)^v$ to denote the iterative sequence, which converges to $\tilde{\sigma}_w^2$. Then we are more or less solving equation

$$\int_\Omega \frac{\sigma_w^2-3|w^{v+1}|^2}{|w^{v+1}|^2+\tilde{\sigma}_w^2}\mathrm{dx}=0 \tag{37}$$

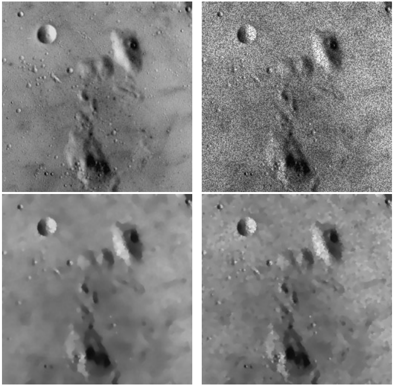
for v being big enough. As $\tilde{\sigma}_w^2$ becomes stable, we may approximate the denominator by $|w^{v+1}|^2+(\sigma_w^2)^v$. And thus, we get that

$$(\sigma_w^2)^{v+1}=\frac{3\int_\Omega \frac{|w^{v+1}|^2}{|w^{v+1}|^2+(\sigma_w^2)^v}\mathrm{dx}}{\int_\Omega \frac{1}{|w^{v+1}|^2+(\sigma_w^2)^v}\mathrm{dx}}. \tag{38}$$

Image denoising: Test1

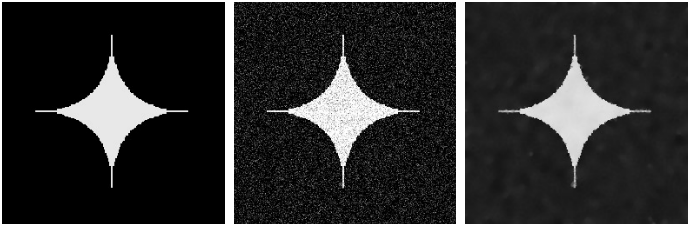
Fg: 3. Experiment of image denoising. (a) Original cameraman image. (b) Noisy image, obtained by adding MPC noise to (a). (c) Recovered image obtained by proposed algorithm with $\alpha > 2$. (d) Recovered image obtained by proposed algorithm with $\alpha \leq 2$.

Image denoising: Test2



Fg: 4. Experiment of image denoising. (a) Original moon image. (b) Noisy image, obtained by adding Poisson noise to (a). (c),(d) Restored images derived by proposed algorithm with different K_d values.

Image denoising: Test3



Fg: 5. Experiment of image denoising. (a) Clean astroid image. (b) Noisy image, obtained by adding truncated Gaussian noise to (a). (c) Restored image derived by proposed algorithm.

BID: Lévy-stable distributions

Lévy-stable distributions such as **Cauchy and Gaussian distributions** play a significant role in radiograph deblurring and denoising.

We propose a combined **2-dimensional Square Cauchy-Gaussian distribution** with PDF

$$B_k(\mathrm{x};\Theta_B) = \sum_{i=1}^{2} \gamma_i p_i(\mathrm{x};\sigma_i^2), \mathrm{x} \in \Omega \tag{39}$$

as prior structure of the kernel, where p_1 and p_2 are defined by

$$p_1(\mathrm{x};\sigma_1^2) = \frac{\sigma_1^2}{\pi(\sigma_1^2 + |\mathrm{x}|^2)^2}, \ \mathrm{x} \in \Omega, \tag{40}$$

and

$$p_2(\mathrm{x};\sigma_2^2) = \frac{1}{2\pi\sigma_2^2} \exp\left(-\frac{|\mathrm{x}|^2}{2\sigma_2^2}\right), \ \mathrm{x} \in \Omega, \tag{41}$$

respectively, the denotation Θ_B represents the set of parameters $\gamma_1, \gamma_2, \sigma_1^2, \sigma_2^2$. $\gamma_i \geq 0$ is a mixture ratio satisfying $\gamma_1 + \gamma_2 = 1$.

Prior density of the blur kernel

We then utilize the KL divergence to measure the difference between the unknown kernel k and the basel structure $B_k(\mathrm{x};\Theta_B)$, and define a prior constraint on the blur kernel k as follows:

$$P_K(k) \propto e^{-J_K(k)}, \tag{42}$$

where the functional $J_K(k)$ is formulated by

$$J_K(k)(\mathrm{x}) = k(\mathrm{x})[\ln k(\mathrm{x}) - \ln B_k(\mathrm{x};\Theta_B) - 1] + B_k(\mathrm{x};\Theta_B). \tag{43}$$

Remark

Obviously, $B_k(\mathrm{x};\Theta_B)$ *is a positive symmetric kernel satisfying*

$$\int_{\mathbb{R}^2} B_k(\mathrm{x};\Theta_B)\mathrm{dx} = 1.$$

If $\gamma_2 \equiv 0$, B_k *is reduced to a pure square Cauchy density function. By the definition of* $J_K(k)$*, it is easy to check that* $J_K(k)$ *is non-negative, convex for* $k > 0$*, and moreover, attains minimum zero at* $k = B_k(\mathrm{x};\Theta_B)$*.*

Infimal convolution-MAP framework

Using Bayes' rule and independence assumption of the random variables U, K, and W, we then resort to the joint Bayesian framework to pose the maximum a posteriori problem

$$\begin{aligned}(\hat{u},\hat{k},\hat{w}) &= \arg\max_{(u,k,w)} P(u,k,w|f) = \arg\max_{(u,k,w)}\{P(f|u,k,w)P(u,k,w)\} \\ &= \arg\max_{(u,k,w)}\left\{P_Z^{U,K}(f-w)P_W(w)P_U(u)P_K(k)\right\} \qquad (44)\end{aligned}$$

for given f.

Following a routine procedure, we can obtain the following raw problem

$$\min_{u,k,w,\Theta}\left\{E(Bu,u,k,w,\Theta)\equiv\tilde{\Psi}(Bu,w)+\Phi(w,\sigma_w^2)+R(\nabla u,\nabla^\alpha u)+S(k,\Theta_B)\right\} \qquad (45)$$

MAP based model

with

$$\tilde{\Psi}(Bu, w) = \int_\Omega \left(\frac{|Bu + b + w - f|^2}{2(Bu + b)} - \beta \ln(Bu + b) \right) \mathrm{dx}, \tag{46}$$

and

$$\Phi(w, \sigma_w^2) = \int_\Omega \left(2\ln(\sigma_w^2 + |w|^2) - \frac{3}{2}\ln(\sigma_w^2) \right) \mathrm{dx}, \tag{47}$$

$$S(k, \Theta_B) = \int_\Omega k(\mathrm{x}) \left[\ln k(\mathrm{x}) - \ln B_k(\mathrm{x}; \Theta_B) - 1 \right] \mathrm{dx}, \tag{48}$$

$$R(\nabla u, \nabla^\alpha u) = \int_\Omega g_1 |\nabla u| \mathrm{dx} + \int_\Omega g_2 |\nabla^\alpha u| \mathrm{dx}. \tag{49}$$

Remark

$S(k, \Theta_B)$ is just a variant of the KL divergence from $k(\mathrm{x})$ to $B_K(\mathrm{x}; \Theta_B)$:

$$D_{KL}(k, B_k) = \int_\Omega \left[k(\mathrm{x}) \ln\left(\frac{k(\mathrm{x})}{B_k(\mathrm{x})} \right) - k(\mathrm{x}) + B_k(\mathrm{x}) \right] \mathrm{dx}. \tag{50}$$

Obviously, the KL divergence is convex for $k > 0$.

Modifications with relaxation

- Due to the allowance of both k and u being unknown variables in (45), the convexity of functional $E(Bu, u, \cdot)$ is destroyed, and this makes the joint MAP estimation difficult.
- Due to the non-commutativity of the Log-Sum operation in the functional $S(k, \Theta_B)$, the parameters in Θ_B are very complicated to optimize or calculate directly.
- To mitigate these drawbacks and gain solvability and efficiency, we then introduce some modifications W.R.T. the components $\tilde{\Psi}(Bu, w)$ and $S(k, \Theta_B)$.

Local estimation

Noting the convexity of the functional $\tilde{\Psi}$ W.R.T. the variable Bu, we introduce an intermediate variable η to approximate the blurry image Bu, and define an equivalent functional as follows

$$\bar{\Psi}(u,k,\eta,w)=\int_\Omega\left(\frac{|\eta+b+w-f|^2}{2(\eta+b)}-\beta\log(\eta+b)\right)\mathrm{dx}+\frac{\varepsilon}{2}\int_\Omega\frac{|k*u-\eta|^2}{2(\eta+b)}\mathrm{dx}, \tag{51}$$

where $\varepsilon>0$ is a punishing parameter. Suppose we have u^v and k^v at the vth iteration [Liu,Gu,Meng,Lu]. Introduce denotations

$$H(u,k)=u^v*k+k^v*u,\quad \tilde{f}^v=u^v*k^v. \tag{52}$$

Substituting the local approximation $H(u,k)-\tilde{f}^v$ for the blurry image $k*u$ in (51), we then obtain an **alternative functional** of the form

$$\begin{aligned}\Psi(u,k,\eta,w;u^v,k^v) \quad = \quad & \int_\Omega\left(\frac{|\eta+b+w-f|^2}{2(\eta+b)}-\beta\log(\eta+b)\right)\mathrm{dx}+\\ & \frac{\varepsilon}{2}\int_\Omega\frac{|H(u,k)-\eta-\tilde{f}^v|^2}{\eta+b}\mathrm{dx}.\end{aligned} \tag{53}$$

EM algorithm for calculating Θ_B

Noting the integral term $\int_\Omega -\ln(B_k(\mathrm{x};\Theta_B))\mathrm{dx}$ in $S(k,\Theta_B)$ is non other than the negative Log-likelihood function of the combined Square Cauchy-Gaussian distribution, we utilize the EM algorithm [Liu,Zhang,Huang,Huan] by introducing a vector-valued auxiliary variable $\phi : \Omega \to [0,1]^2$ with elements (ϕ_1, ϕ_2) satisfying

$$\phi \in \Delta = \left\{ \phi(\mathrm{x}) : 0 < \phi_i(\mathrm{x}) < 1, \sum_{i=1}^{2} \phi_i(\mathrm{x}) = 1, \forall \mathrm{x} \in \Omega \right\}, \tag{54}$$

and define an upper bound of the functional as follows:

$$\mathcal{H}(k,\Theta_B,\phi) = \int_\Omega k(\mathrm{x}) \left\{ \ln k(\mathrm{x}) - \sum_{i=1}^{2} \phi_i \ln(\gamma_i p_i(\mathrm{x};\sigma_i^2)) + \sum_{i=1}^{2} \phi_i(\mathrm{x}) \ln \phi_i(\mathrm{x}) - 1 \right\} \mathrm{dx}. \tag{55}$$

BID model

For concision, we introduce denotation

$$\Pi \doteq \{u, k, \eta, w, \Theta, \phi\}.$$

Substituting $\Psi(u, k, \eta, w; u^v, k^v)$ and $\mathcal{H}(k, \Theta_B, \phi)$ for $\tilde{\Psi}(Bu, w)$ and $S(k, \Theta_B)$ in (46) and (48), respectively, we then propose the following hybrid regularization model

$$\min_{\Pi} \{\mathcal{E}(\Pi) \models \Psi(u, k, \eta, w; u^v, k^v) + R(\nabla u, D^\alpha u) + \mathcal{H}(k, \Theta_B, \phi) + \Phi(w, \sigma_w^2)\} \tag{56}$$

to address our Poisson BID problem.

$\Theta = \Theta_B \cup \{\sigma_w^2\}$ denotes the set of unknown parameters in our model.

Convexity

Proposition

Let Ω *be an open bounded subset of* $\mathbb{R}^2$ *with* $|\Omega| = \int_\Omega 1\mathrm{dx}$. *Assume that* $\sigma_w^2 > 0$, $b > 0$, $\eta \geq 0$, $\phi = \{\phi_1, \phi_2\} \in \Delta$. *Density function* p_i *is defined by (40) with* $\sigma_i^2 > 0$, $\gamma_i \geq 0$, $i = 1, 2$. g_1 *and* g_2 *are continuous functions on* Ω, *which are bounded below from zero.* $w \in L^1(\Omega) \cap L^4(\Omega)$, $0 \leq u \in BV_g^\alpha(\Omega)$ *[wei,kong]. Also assume that* $0 \leq k \in L^1(\Omega)$ *satisfies* $k \ln k \in L^1(\Omega), \int_\Omega k\phi_1 |\mathrm{x}|^4 \mathrm{dx} < +\infty$, $\int_\Omega k\phi_2 |\mathrm{x}|^2 \mathrm{dx} < \infty$. *Then the functional* $\mathcal{E}(u, k, \eta, w, \Theta, \phi)$ *is convex W.R.T. each of the variables* u, k, η, ϕ_i *and* γ_i *as the others are fixed. Moreover, if there hold*

$$\frac{\eta + b}{2} \leq \sigma_w^2 \leq \frac{3 + 2\sqrt{3}}{|\Omega|} \int_\Omega |w|^2 \mathrm{dx}, \tag{57}$$

and

$$0 < \sigma_1^2 \leq \frac{(1 + \sqrt{2}) \int_\Omega k\phi_1 |\mathrm{x}|^2 \mathrm{dx}}{\int_\Omega k\phi_1 \mathrm{dx}}, \quad 0 < \sigma_2^2 \leq \frac{\int_\Omega k\phi_2 |\mathrm{x}|^2 \mathrm{dx}}{2 \int_\Omega k\phi_2 \mathrm{dx}}, \tag{58}$$

then the functional is also convex W.R.T. the variables w, σ_w^2, σ_i^2, *respectively.*

BCD-ADMM based algorithm: blur kernel calculation and image restoration

Given u^v, k^v, η^v, w^v, Θ_B^v, and ϕ^{v+1}. We consider the solution of the primary problem

$$(u^{v+1}, k^{v+1}) = \arg\min_{(u,k)} \left\{ \mathcal{A}(u,k) + R(\nabla u, \nabla^\alpha u) + \mathcal{H}(k, \Theta_B^v, \phi^{v+1}) \right\} \tag{59}$$

with

$$\mathcal{A}(u,k) = \frac{\varepsilon}{2} \int_\Omega \frac{|H(u,k) - \eta^v - \tilde{f}^v|^2}{\eta^v + b} \tag{60}$$

Remark

The functional $\mathcal{A}(u,k)$ can be regarded as a variant of traditional reweighted fidelity corresponding to the mixture of Poisson and Gaussian noises. If $\eta^v \equiv 0$, it is reduced to the single Gaussian case.

Updating u

Introduce denotation $A_k u = k^v * u$. We turn to consider the following quadratic u-subproblem

$$u^{v+1} = \arg\min_u \left\{ \int_\Omega \frac{\varepsilon |A_k u - \eta^v|^2}{2(\eta^v + b)} \mathrm{dx} + \frac{1}{2\triangle t} \int_\Omega |u - u^v|^2 \mathrm{dx} + \int_\Omega \left(\frac{\rho_d}{2} |\nabla u - d^v + \frac{\mu_d^v}{\rho_d}|^2 + \frac{\rho_h}{2} |\nabla^\alpha u - h^v + \frac{\mu_h^v}{\rho_h}|^2 \right) \mathrm{dx} \right\} \quad (61)$$

where the proximal term in u is involved to ensure uniqueness as well as efficiency, and $\triangle t$ is a positive constant regarded as scale parameter. The optimality equation of this subproblem incorporating Neumann boundary conditions is given by

$$u - \triangle t \mathrm{div}(\rho_d \nabla u) = T(u), \quad (62)$$

where the source term $T(u)$ is given by

$$T(u) = u^v - \triangle t \left[\varepsilon A_k^\top \left(\frac{A_k u - \eta^v}{\eta^v + b} \right) + \mathrm{div}(\rho_d d^v - \mu_d^v) + \overline{(-1)^\alpha} \mathrm{div}^\alpha (\rho_h (\nabla^\alpha u - h^v) + \mu_h^v) \right] \quad (63)$$

where $A_k^\top$ is the adjoint operator of A_k.

Updating k

Introduce denotations

$$g^{v+1} = \tilde{f}^v + \eta^v - u^{v+1} * k^v, \quad A_u k = u^v * k,$$

and

$$F(k) = \int_\Omega \frac{|A_u k - g^{v+1}|^2}{2(\eta^v + b)} \mathrm{dx}. \tag{64}$$

We then consider the following minimization problem

$$k^{v+1} = \arg\min_k \left\{ \varepsilon F(k) + \mathcal{H}(k, \Theta_B^v, \phi^{v+1}) \right\}. \tag{65}$$

We consider the expansion of $F(k)$ in a **Taylor series** around k^v, which can be expressed

$$F(k) = F(k^v) + \langle \nabla F(k^v), k - k^v \rangle + \frac{1}{2} \langle k - k^v, \nabla^2 F(k - k^v) \rangle, \tag{66}$$

where the Hessian $\nabla^2 F = A_u^\top (\frac{A_u}{\eta^v + b})$, $A_u^\top$ represents the adjoint operator of A_u, $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. Since $\nabla^2 F$ can be large and dense in FXR imaging, we make the approximations $\nabla^2 F \approx \delta_k I$, I denotes the identity matrix, and

$$F(k) \approx F(k^v) + \langle A_u^\top (\frac{A_u k^v - g^{v+1}}{\eta^v + b}), k - k^v \rangle + \frac{\delta_k}{2} \|k - k^v\|_2^2. \tag{67}$$

Updating k

Exploiting the **Barzilai-Borwein choice for** δ_k, we have

$$\begin{aligned}\delta_k &= \arg\min_{\delta} \left\|\nabla F(k^v) - \nabla F(k^{v-1}) - \delta(k^v - k^{v-1})\right\|_2^2 \\ &= \frac{\|\frac{A_u(k^v - k^{v-1})}{\eta^v + b}\|_2^2}{\|k^v - k^{v-1}\|_2^2} \leq \max_{\|k^v\|_2=1} \|\frac{A_u k^v}{\eta^v + b}\|_2^2 \doteq L_k. \end{aligned} \tag{68}$$

By completing square, the approximate formulation (67) can be rewritten by

$$F(k) \approx F(k^v) - \frac{1}{2\delta_k}\|A_u^\top(\frac{A_u k^v - g^{v+1}}{\eta^v + b})\|_2^2 + \frac{\delta_k}{2}\|k - k^v + \delta_k^{-1} A_u^\top(\frac{A_u k^v - g^{v+1}}{\eta^v + b})\|_2^2. \tag{69}$$

With this replacement in (65), we then utilize the majorization-minimization approach to consider the following minimization problem:

$$\min_k \left\{ \frac{\varepsilon \delta_k}{2} \int_\Omega |k - k^v + \frac{1}{\delta_k} A_u^\top(\frac{u^{v+1} * k^v - \eta^v}{\eta^v + b})|^2 \mathrm{dx} + \mathcal{H}(k, \Theta_B^v, \phi^{v+1}) \right\}, \tag{70}$$

since there holds $A_u k^v - g^{v+1} = u^{v+1} * k^v - \eta^v$.

Updating k: A Majorization-Minimization approach

Introduce denotation

$$\bar{p}_{cg}(\Theta_B^v, \phi^{v+1}) = \sum_{i=1}^{2} \phi_i^{v+1} \ln(\gamma_i^v p_i(\mathrm{x}; (\sigma_i^2)^v)) - \sum_{i=1}^{2} \phi_i^{v+1} \ln \phi_i^{v+1}. \tag{71}$$

To define the upper bound of $\mathcal{H}(k, \Theta_B^v, \phi^{v+1})$, we utilize the Taylor expansion of the logarithm around the vth estimate of k up to the first term, and obtain

$$\psi(k|k^v) = \int_\Omega k[(\ln k^v + \frac{k - k^v}{k^v}) - \bar{p}_{cg}(\Theta_B^v, \phi^{v+1}) - 1]\mathrm{dx}. \tag{72}$$

According to the concavity of the logarithm function, it is easy to check that ψ satisfies the following properties

$$\begin{cases} \psi(k|k^v) & \geq \int_\Omega k[\ln k - \bar{p}_{cg}(\Theta_B^v, \phi^{v+1}) - 1]\mathrm{dx} \\ \psi(k^v|k^v) & = \int_\Omega k^v[\ln k^v - \bar{p}_{cg}(\Theta_B^v, \phi^{v+1}) - 1]\mathrm{dx}. \end{cases} \tag{73}$$

Updating k: A Majorization-Minimization approach

And thus, by plugging ψ in (65), we obtain the following quadratic minimization problem:

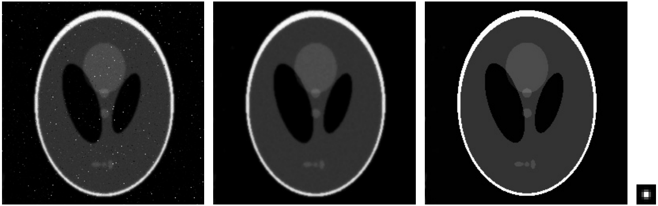
$$\min_k \left\{ \frac{\varepsilon\delta_k}{2} \|k - k^v + \frac{1}{\delta_k} A_u^\top \left(\frac{u^{v+1} * k^v - \eta^v}{\eta^v + b}\right)\|_2^2 + \int_\Omega k\left[\frac{k - k^v}{k^v} + \ln k^v - \bar{p}_{cg}(\Theta_B^v, \phi^{v+1}) - 1\right] \mathrm{dx} \right\}. \tag{74}$$

By direct computation, we obtain the following formulation

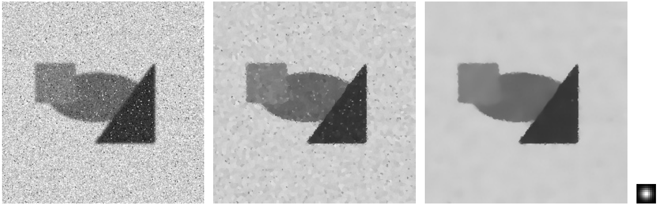
$$\bar{k}^{v+1} = k^v \left[1 + \frac{\bar{p}_{cg}(\Theta_B^v, \phi^{v+1}) - \ln k^v - \varepsilon A_u^\top \left(\frac{u^{v+1} * k^v - \eta^v}{\eta^v + b}\right)}{2 + \varepsilon\delta_k k^v} \right] \tag{75}$$

for updating the blur kernel.

Numerical tests

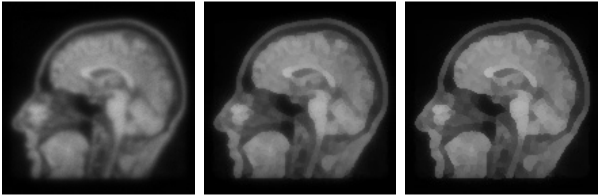


Fg: 6.Blind image deconvolution: (**a**) A noisy blurred image (SSIM=0.4581). (**b**) u^0, obtained by median filter (SSIM=0.7275). (**c**) A restored image using our proposed Algorithm 1 (SSIM=0.9407) with $K_d = 0.0075$. (**d**) The restored blur kernel.



Fg: 7.Blind image deconvolution: (**a**) A noisy blurred image (SSIM=0.3118). (**b**) u^0, obtained by performing MPsC-$TV^{1+\alpha}$ model and then enhancing by L-filter (SSIM=0.6228). (**c**) A restored image using our proposed Algorithm 1 (SSIM=0.9819). (**d**) The restored blur kernel.

Numerical tests



Fg: 8.Blind image deconvolution: (**a**) A noisy blurred MRI image. (**b**) A restored image using our proposed Algorithm 1 with $K_d = 0.0025$. (**c**) A restored image using our proposed Algorithm 1 with $K_d = 0.0015$.

Conclusion

- **Square Cauchy distribution** was introduced to address the fundamental problems such as denoising and deblurring in radiograph processing.
- Two new mixture models, including **mixed Poisson-(1D) Square Cauchy distribution**, and **combined (2D) Square Cauchy-Gaussian distribution**, were utilized to characterize the noise and system blur in NR or FXR.
- Two multi-convex optimization frameworks along with BCD-ADMM based algorithms were proposed to address the denoising and blind deblurring problems.

Thank You for Your attention!