Ambiguity of fuzzy measure and inner dependency matrix in AHP from a viewpoint of sensitivity analysis

Shin-ichi Ohnishi, Takahiro Yamanoi
Hokkai-Gakuen University, Japan
Hokkaido University, Japan

Introduction

Analytic Hierarchy Process (AHP) methodology is a very convenient and popular in the multi criterion decision making field.

- ✓ Criteria must be independent perfectly, because additive measure weight
- ✓ Data matrix must have enough consistency for its reliability.

HOWEVER, it is very hard in practice

- ➢ Perfect independence among criteria in the hierarchical structure.
- ➢ Enough consistency in the data matrix.

Extended methods

- fuzzy measure AHP
- Inner dependence AHP

In this study

- ◆ **Consider how to treat ambiguity or vagueness in these two extension methods, and compare these two methods from a viewpoint of their sensitivity analysis.**

Fuzzy Measure AHP (Ichihashi 1989)

- An extension of the Normal AHP
- **Using fuzzy measure as non-additive weight**
- **Employing Choquet integral for aggregating total priority.**
- Two types of decision by use of non-additive fuzzy measure
  - Substitutive decision (possibility measure)
  - Complementary decision (necessity measure)

- The reversals of the priority order due to the addition of a similar activity

Overall weights of fuzzy AHP

<Substitutive decision>

The upper limit expectation based on possibility measure

$$y_p^{(\mathrm{Pl})} = \sum_{l}^{q} m(A_l) \max_{x_i \in A_l} f_p(x_i)$$

<Complementary decision>

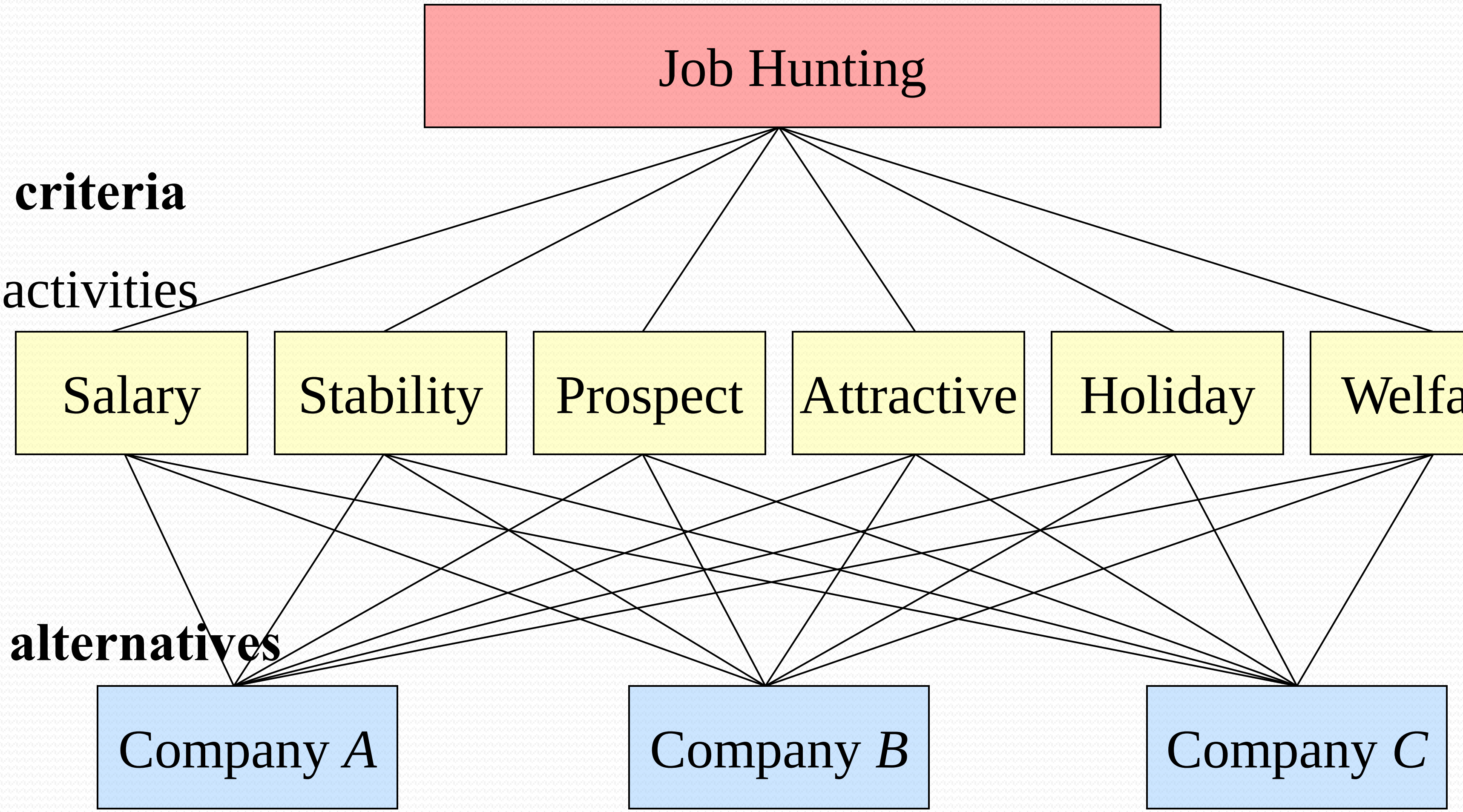
The lower limit expectation based on necessity measure

$$y_p^{(\mathrm{Bel})} = \sum_{l}^{q} m(A_l) \min_{x_i \in A_l} f_p(x_i)$$

$f_p(x_i)$: weights of p-th alternative with respect to x_i

Hierarchy structure

1. <u>**Representation by a hierarchy**</u>
2. Pairwise comparison matrices
3. (Consistency check)
4. Local weights of criteria
5. Global weights of alternative



Example: Fuzzy measure AHP

(P2)pairwise comparison matrix

	Salary	Stability	Prospect	Attractive	Holiday	Welfare
Salary	1	1/5	1/5	1/5	1/2	1/3
Stability		1	3	4	7	5
Prospect			1	3	6	5
Attractive				1	7	3
Holiday					1	1/5
Welfare						1

(P4)weights of activities

$$\begin{pmatrix} \text{Salary} \\ \text{Stability} \\ \text{Prospect} \\ \text{Attractive} \\ \text{Holiday} \\ \text{Welfare} \end{pmatrix} = \begin{pmatrix} 0.04 \\ 0.41 \\ 0.26 \\ 0.16 \\ 0.04 \\ 0.09 \end{pmatrix}$$

(max become 1 and sort)

$$\begin{pmatrix} \text{Stability} \\ \text{Prospect} \\ \text{Attractive} \\ \text{Welfare} \\ \text{Holiday} \\ \text{Salary} \end{pmatrix} = \begin{pmatrix} 1.00 \\ 0.63 \\ 0.39 \\ 0.22 \\ 0.10 \\ 0.10 \end{pmatrix}$$

basic probability assignment by difference

$A_5 = \{\text{Sta}\}$ $\quad m(A_5) = 0.37$

$A_4 = \{\text{Sta, Pro}\}$ $\quad m(A_4) = 0.24$

$A_3 = \{\text{Sta, Pro, Att}\}$ $\quad m(A_3) = 0.17$

$A_2 = \{\text{Sta, Pro, Att, Wel}\}$ $\quad m(A_2) = 0.12$

$A_1 = \{\text{Sta, Pro, Att, Wel, Hol, Sal}\}$ $\quad m(A_1) = 0.10$

Example: Fuzzy measure AHP

Basic probability assignment

$$m(A_5) = 0.37 \qquad A_5 = \{\text{Sta}\}$$
$$m(A_4) = 0.24 \qquad A_4 = \{\text{Sta, Pro}\}$$
$$m(A_3) = 0.17 \qquad A_3 = \{\text{Sta, Pro, Att}\}$$
$$m(A_2) = 0.12 \qquad A_2 = \{\text{Sta, Pro, Att, Wel}\}$$
$$m(A_1) = 0.10 \qquad A_1 = \{\text{Sta, Pro, Att, Wel, Hol, Sal}\}$$

Subsets of focal element

(P4)Local weights of alternative$_p$

	Company A
Stability	0.121
Prospect	0.180
Attractive	0.070
Welfare	0.121
Holiday	0.157
Salary	0.158

$$y_p^{(\mathrm{Pl})} = \sum_{l}^{q} m(A_l) \max_{x_i \in A_l} f_p(x_i)$$
$$= 0.158$$

$$y_p^{(\mathrm{Bel})} = \sum_{l}^{q} m(A_l) \min_{x_i \in A_l} f_p(x_i)$$
$$= 0.101$$

Inner dependence AHP

Saaty 1991

By dependency matrix $F=(f_{ij})$,
modified weights $\boldsymbol{w}^{(n)}$ considering dependency relation

$$\boldsymbol{w}^{(n)} = F\boldsymbol{w}$$

- $\boldsymbol{w}$ is weight vector assuming independency among elements (weight of normal AHP)
- F is determined by eigen vectors of influence matrix.

Fuzzy theory

Fuzzy Set

「old vase」

This pot *x*: 100 years old
(crisp)

Set *A*: old vase:
not determined how old
(fuzzy)

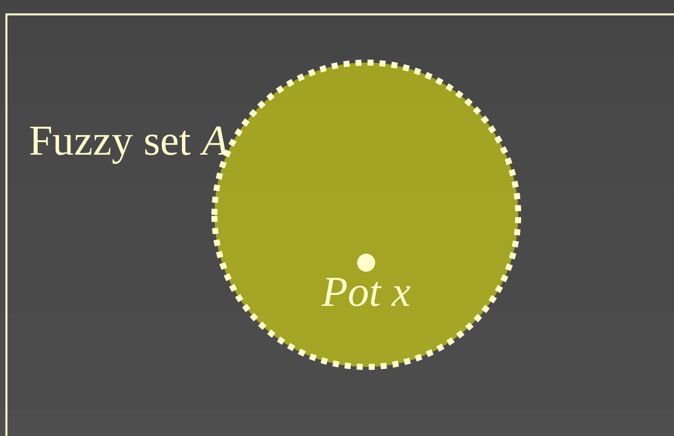
Fuzzy Measure

「old vase」

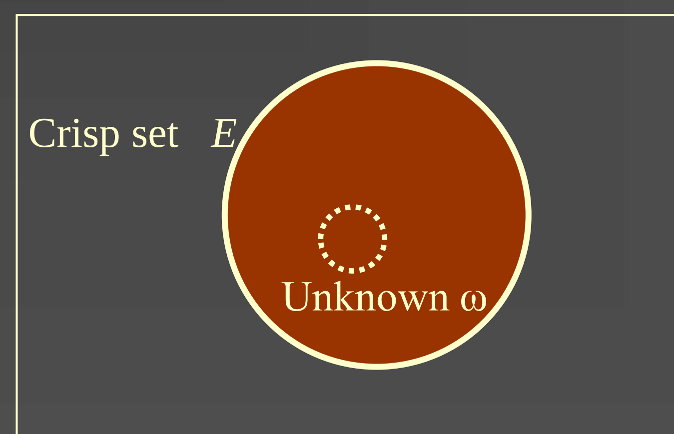
This pot ω: unknow how old
（fuzzy）

Set *E*: old vase :
over 100 years old
（crisp）

Vagueness



Ambigyuity



Fuzzy measure AHP

- Fuzziness to resolve is "ambiguity" because it is about fuzzy measure
- It is difficult for decision maker to understand results.
  - ✓ Weights of subsets but not of each criterion (elements)
  - ✓ Fuzzy integral as aggregation.
- Deep knowledge of experts is not necessary

Inner dependence AHP

- Fuzziness to resolve is "ambiguity" because it is about dependency among criteria
- It is easy for decision maker to understand the method.
- Deep knowledge of experts is necessary, furthermore it is difficult estimate exact influence.
  - ✓ "vagueness" may remain.

Steps of sensitivity analysis

（Ohnishi *et al* 1997）

(i) Giving perturbation $\varepsilon a_{ij} d_{ij}$ to each element a_{ij} of matrix A.

(ii) Representing fluctuation of consistency or weight by linear combination of d_{ij}.

(iii) Estimating amount of influence by coefficient of d_{ij}.

$$A(\varepsilon) = A + \varepsilon D_A$$

$A = a_{ij} \ (i, j = 1, \ldots, n)$

perturbation $D_A = \left(a_{ij} d_{ij} \right)$

Sensitivity analysis of ***weight in normal AHP***

Corollary 2 (Ohnishi et al. 1997)

The weight of the perturbed comparison matrix

$$\boldsymbol{w}(\varepsilon) = \boldsymbol{w} + \varepsilon \sum_{i}^{n} \sum_{j}^{n} h_{ij}^{(k)} d_{ij} + \boldsymbol{o}(\varepsilon)$$

$h^{(k)}_{ij}$:const.

Sensitivity analysis of ***weight*** in ***fuzzy measure AHP***

Theorem2 (substitutive decision)

$y^{(\mathrm{pl})}{}_p$: overall weight of the p-th alternative based on A

$y^{(\mathrm{Pl})}{}_p(\varepsilon)$: overall weight of the p-th alternative based on peterbed data matrix $A(\varepsilon)$

($p = 1,\ldots,m$)

$$y_p^{(\mathrm{Pl})}(\varepsilon) = y_p^{(\mathrm{Pl})} + \varepsilon \sum_{i,j}^{n} \left\{ \sum_{l}^{q} (h_{ij}^{l} - h_{ij}^{l-1}) \max_{x_i \in A_l} f_p(x_i) \right\} d_{ij} + o(\varepsilon)$$

Viewpoint of their sensitivity analysis

Fuzzy measure AHP

- Not for weights but for basic probability assignment.
- Useful for total weight of alternatives

Inner dependence AHP

- Not only analysis for local weights but also it for dependency matrix from influence matrix.
- Useful for modified weight of criteria

Summary

- Consider how to treat ambiguity or vagueness in these two extension methods, and compare these two methods from a viewpoint of their sensitivity analysis.
- In two extended AHP, fuzziness to resolve is “ambiguity”
- For decision maker inner dependence AHP is easy to understand.
- it is difficult estimate exact influence, “vagueness” may remain in inner dependence method.
- Deep knowledge of experts is not necessary in fuzzy measure AHP

from a viewpoint of sensitivity analysis

- Sensitivity analysis for weight is also useful for basic probability assignment total weight of alternatives in fuzzy measure.
- Results are useful for modified weight of criteria and dependency matrix in inner dependence AHP.