Reduction of Decoherence in Quantum Information Systems Using Direct Adaptive Control of Infinite Dimensional Systems

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Motivation: Quantum Computing

“A Quantum computer will operate differently from a Classical one. It will be involved w physical systems on an atomic scale, eg atoms, photons, trapped ions, or nuclear magnetic moments”

… R. Feynman 40 years ago

Decoherence is the loss of information from a system into the environment. Entanglements are generated between the system and environment, which have the effect of sharing quantum information with—or transferring it to—the surroundings

Reduced with Infinite Dimensional Direct Adaptive Control
(And Quantum Error Correction)
Quantum Basics
(Dirac & Von Neumann)

Observable $A : X \rightarrow X$

$Ax = \sum_{k=1}^{\infty} \lambda_k (x, \phi_k) \phi_k = \sum_{k=1}^{\infty} \lambda_k P_k x \& \sigma(A) \equiv \{ \lambda_1, \lambda_2, \lambda_3, \ldots \}$

Pure States: $\phi_k$ eigenfunctions of $A$

State $\phi \in X$ complex infinite-dimensional separable Hilbert Space:

$(\phi, \phi) = 1$ or $\|\phi\| = 1 \Rightarrow \phi = \sum_{k=1}^{\infty} c_k \phi_k \& \|\phi\|^2 = \sum_{k=1}^{\infty} |c_k|^2 = 1$

$\therefore$ "A (mixed) state is a linear combination of pure states"

Special Case: Quantum SPIN Systems are FINITE Dimensional
Dynamics: Schrodinger Wave Equation

\[ \phi \in X \text{ complex Hilbert Space} \]

\[ i\hbar \frac{\partial \phi}{\partial t} = H_0 \phi \]

Hamiltonian Energy Operator

Discrete Spectrum \( \sigma(H_0) = \{\lambda_k\}_{k=1}^{\infty} \)

\[ \Rightarrow \phi(t) = \underbrace{U_0(t)}_{\text{Unitary Group}} \phi(0) = e^{\frac{-i}{\hbar} H_0 t} \phi(0) = \sum_{k=1}^{\infty} e^{\frac{-i\lambda_k}{\hbar} t} (\phi(0), \phi_k)\phi_k \text{ with } (\phi_k, \phi_l) = \delta_{kl} \]

\[ \Rightarrow \|\phi(t)\|^2 = \text{Probability Distribution for the Energy} \]

in the Quantum State \( \phi(t) \Rightarrow \|\phi(t)\| = \|\phi(0)\| \)

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in the Quantum State \( \phi(t) \)

\[ \Rightarrow \|\phi(t)\| = \|\phi(0)\| \]
The interpretation of Quantum Measurement is still a controversial part of Quantum Theory.

**Entanglement**

\[ X = X_S \otimes X_M \]
\[ \phi = \sum_{k,l} \alpha_{kl} (\phi^S_k \otimes \phi^M_l) \neq h \otimes w \]

A quantum measurement is an entanglement with the environment (measuring device).
“Simplicity” via Infinite Dimensional Spaces

\[
\begin{align*}
\frac{\partial x}{\partial t} &= Ax + Bu + \Gamma u_D; A \text{ generates a } C_0 \text{-semigroup } U(t) \\
Bu &= \sum_{i=1}^{M} b_i u_i \\
x(0) &= x_0 \in D(A) \subset X \\
y &= Cx = [(c_1, x)(c_2, x) \ldots (c_m, x)]^*; b_i, c_j \in D(A) \\
\Rightarrow x(t, w_0) &= U(t)x_0; \forall t \geq 0
\end{align*}
\]

\[C_0 \text{-Semigroup of Bounded Operators } U(t) : \]
\[
\begin{align*}
U(t+s) &= U(t)U(s) \text{ (semigroup property)} \\
\frac{d}{dt}U(t) &= AU(t) = U(t)A \text{ ( } A \text{ generates } U(t)) \\
U(t)x_0 \xrightarrow{t \to 0} x_0 \text{ (continuous at } t = 0) 
\end{align*}
\]


Semigroups

Closed Linear Operator

\[
\begin{aligned}
\frac{\partial x}{\partial t} &= Ax \\
x(0) &= x_0 \in D(A)
\end{aligned}
\]

t \Rightarrow x(t) = U(t)x_0

\[\text{dim } X < \infty \Rightarrow U(t) = e^{At} = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}\]

\[C_0 \text{ -- Semigroup}\]

\[U(t) : X \to X \text{ bounded operators } t \geq 0\]

\[\text{Generator} : Ax = \lim_{t \to 0^+} \frac{U(t)x - x}{t} \text{ with } D(A) = \{x / \lim_{t \to 0^+} \text{ exists } \} \text{ dense in } X\]

LaPlace Transform

\[\begin{aligned}
L(U(t)) &= (\lambda I - A)^{-1} \equiv R(\lambda, A) \text{ Resolvent Operator} \\
L^{-1}(R(\lambda, A)) &= U(t)
\end{aligned}\]
Spectrum of $A$

Resolvent Set $\rho(A) \equiv \{ \lambda / R(\lambda, A) : X \to X \text{ bounded linear op on } X \}$

Spectrum $\sigma(A) \equiv \rho(A)^c = \sigma_{\text{point}}(A) \cup \sigma_{\text{cont}}(A) \cup \sigma_{\text{residual}}(A)$

$\sigma_{\text{point}}(A) \equiv \{ \lambda / \lambda I - A \text{ is NOT 1-1} \} = \{ \lambda / \exists \phi \neq 0 \exists \lambda \phi = A\phi \}$

$\sigma_{\text{cont}}(A) \equiv \{ \lambda / \lambda I - A \text{ is 1-1, but its range is only dense in } X \}$

$\sigma_{\text{residual}}(A) \equiv \{ \lambda / \lambda I - A \text{ is 1-1, but range is a proper subspace of } X \}$

**Theorem (Gearhart, Pruss, & Greiner):**

Assume $A$ generates a $C_0$-semigroup $U(t)$ on a Hilbert space $X$.

$U(t)$ is exponentially stable $\iff \Re \lambda > 0 \iff \lambda \in \rho(A)$ and $\| R(\lambda, A) \| \leq M < \infty$, for all such complex $\lambda$. 
**When is a Semigroup Exponentially Stable?**

*Lumer–Phillips* (Renardy & Rogers 1993):

\[ D(A) \text{ dense in complex Hilbert space } X, \]

\[ \text{Re}(Ax, x) \leq -\alpha \|x\|^2 \quad \forall x \in D(A), \alpha > 0 \]

and

\[ \exists \text{ real } \lambda^* > -\alpha \in A - \lambda^* I \text{ is onto} \]

\[ \Rightarrow \|U(t)\| \leq e^{-\alpha t}; t \geq 0 \text{ (exponentially stable semigroup)} \]

**Theorem** (Gearhart, Pruss, & Greiner):

Assume \( A \) generates a \( C_0 \)-semigroup \( U(t) \) on a Hilbert space \( X \).

\( U(t) \) is exponentially stable \( \iff \) \( \text{Re } \lambda > 0 \Rightarrow \lambda \in \rho(A) \) and

\[ \|R(\lambda, A)\| \leq M < \infty, \text{ for all such complex } \lambda \]
Direct Adaptive Model Following Control
(Wen-Balas 1989)

The Godfather: K.S.Narendra

Plant

Reference Model

Infinite Dimensional

Adaptive Gain Laws

\((u_m, x_m, e_y)\) Known Signals

\(y_m\)

\(y\)

\(e_y\)

\(t \to \infty \to 0\)
Direct Adaptive Persistent Disturbance Rejection (Fuentes-Balas 2000)

Reference model

Disturbance Generator

Plant

Adaptive Gain Laws

Signals Known, Basis Disturbance

Disturbance Generator

$u_m \rightarrow y_m$

$u \rightarrow y$

$e_y \rightarrow 0$

$(u_m, x_m, e_y, \phi_2)$ Known Signals

Disturbance Basis
Direct Adaptive Control Theory
Is Not Complicated!

Adaptive Regulation

Infinite - Dimensional Plant

\[
\begin{align*}
\frac{dx}{dt} &= Ax + Bu \\
y &= Cx; x_0 \in D(A) \subset X
\end{align*}
\]

Controller

Nonlinear Adaptive Gain Law

\[
\dot{G} = -yy^* \sigma; \sigma > 0
\]

Use ONLY Outputs & Know Almost NOTHING about the Plant
For Finite & Infinite Dimensions

All Roads Lead To Rome

\[
\begin{aligned}
\dot{x} &= Ax + Bu = Ax + \sum_{i=1}^{m} b_i u_i \\
x(0) &= x_0 \in D(A) \subset X \\
y &= Cx = [(c_1, x) \quad (c_2, x) \quad \ldots \quad (c_m, x)]^T
\end{aligned}
\]

with \((A, B, C)\) Almost Strictly Dissipative (ASD)

\[\Rightarrow\] Adaptive Controller

\[
\begin{aligned}
\dot{u} &= G(t) y \\
\dot{G}(t) &= -yy^* \sigma; \sigma > 0
\end{aligned}
\]

produces \(x(t) \xrightarrow{t \to \infty} 0\)

with bounded adaptive gains \(G(t)\)
Finite- Dimensional LINEAR ASD: Two Simple Open-Loop Properties

High Frequency Gain is Sign-Definite (CB > 0)

Open-Loop Transfer Function is Minimum Phase (i.e. Transmission Zeros are all stable)

Almost Strictly Dissipative

Adaptive Regulation \[ \begin{cases} u = Gy \\ \dot{G} = -yy^* \sigma; \sigma > 0 \end{cases} \]

produces \( x(t) \xrightarrow{t \to \infty} 0 \) with bounded adaptive gains \( G(t) \)
Our Infinite-Dimensional Version of the “Two Simple Open Loop Properties” Theorem

\[
\begin{align*}
\frac{\partial x}{\partial t} &= Ax + Bu = Ax + \sum_{i=1}^{m} b_i u_i; A \text{ generates a } C_0 \text{ semigroup} \\
x(0) &= x_0 \in D(A) \subset X \\
y &= Cx = [(c_1, x) \ (c_2, x) \ldots (c_m, x)]^*; b_i, c_j \in D(A)
\end{align*}
\]

**Theorem**: Def: \( \lambda_* \in C \) is a transmission zero of \((A, B, C)\) when \( N(H(\lambda_*)) \neq \{0\} \)

where \( H(\lambda) = \begin{bmatrix} A - \lambda I & B \\ C & 0 \end{bmatrix} : D(A) \mathbb{R}^M \rightarrow X \mathbb{R}^M \) closed linear operator

\((A, B, C)\) is Almost Strictly Dissipative if and only if

\[
CB = [(c_j, b_i)]_{m \times m} > 0 \quad \text{and} \quad \text{Transmission Zeros}(A, B, C) \equiv \{ \lambda \mid N(H(\lambda)) \neq \{0\} \} = \sigma_p(\overline{A_{22}}) \quad \text{"stable"}
\]

(i.e., \( \overline{A_{22}} \) generates exponentially stable semigroup)

Adaptive Model Tracking in the Presence of Persistent Disturbances

Reference Model

\[
\begin{align*}
\dot{x}_m &= A_m x_m + B_m u_m \\
y_m &= C_m x_m
\end{align*}
\]

Controller

Adaptive Gain Law

\[
\dot{G} = \begin{bmatrix} \dot{G}_e & \dot{G}_u & \dot{G}_m & \dot{G}_D \end{bmatrix} = -h(e_y, u_m, y_m, \phi_D)
\]

\[u_D = \theta L \phi_D\]

\[\Delta x \equiv x - x^* \xrightarrow{t \to \infty} 0\]

Can also be Infinite Dimensional

\[e_y \xrightarrow{t \to \infty} 0\]
Adaptive Control Law

\[ u = G_u u_m + G_m x_m + G_D \phi_D + G_e e_y \]

where

\[
\begin{align*}
\dot{G}_u &= -e_y u_m^* \sigma_u; \sigma_u > 0 \\
\dot{G}_m &= -e_y x_m^* \sigma_m; \sigma_m > 0 \\
\dot{G}_D &= -e_y \phi_D^* \sigma_D; \sigma_D > 0 \\
\dot{G}_e &= -e_y e_y^* \sigma_e; \sigma_e > 0 
\end{align*}
\]

Gain Adaptation Laws
Adaptive Quantum Model Tracking to Reduce Decoherence

Reference Model: Closed System

Desired Hamiltonian $H_*$

Open Physical System

$u_D = \theta L \phi_D$

$y = (c, \phi)$

$\frac{ih}{\partial t} \frac{\partial \varphi}{\partial t} = \varphi + H \left( u \right) \varphi + H_{\text{Linblad}} \varphi$

$y = \left( c, \varphi \right)$

QND Measurement: Quantum Filtering With Unitary Operators and Projections

$e_y \equiv y - y_m \rightarrow 0$

Adaptive Quantum Controller

Adaptive Gain Law

$\dot{G} = \left[ \dot{G}_e \dot{G}_u \dot{G}_m \dot{G}_D \right] = -h(e_y, u_m, y_m, \phi_D)$
Schroedinger Equation Control

\[ i\hbar \frac{\partial \phi}{\partial t} = H_0 \phi + H_C \phi \]

- **Self-Adjoint Energy Hamiltonian**
- **Feedback Control Hamiltonian**

\[ \begin{cases} \frac{\partial \varphi}{\partial t} = A\varphi + Bu = A\varphi + \sum_{i=1}^{m} b_i u_i; A \equiv -\frac{i}{\hbar} H_0 \\ y = C\varphi = [(c_1, \varphi) (c_2, \varphi) \ldots (c_m, \varphi)]^* \end{cases} \]

Adaptive Controller: \( u = G(t) y \& u_i = \sum_{j=1}^{m} g_{ij}(t)(c_j, x) \)

\[ \Rightarrow \frac{\partial}{\partial t} \varphi = A\varphi + \sum_{i=1}^{m} b_i u_i = A\varphi + \sum_{i=1}^{m} \sum_{j=1}^{m} b_i g_{ij}(t)(c_j, \varphi) \]

\[ \Rightarrow \varphi(t, x_0) = U(t)\varphi_0 ; \forall t \geq 0 \text{ (Not Unitary)} \]
Quantum Adaptive Control Approach

\[
\begin{align*}
    x_N & \equiv P_N x \xrightarrow{t \to \infty} 0 \quad \text{exponentially} \\
    \| x_R \| & \equiv \| P_R x \| = \| P_R x_0 \| \quad \text{constant} \\
    P_N x & \equiv \sum_{k=1}^{N} (\phi_k, x) \phi_k \\
    P_R x & \equiv \sum_{k=N+1}^{\infty} (\phi_k, x) \phi_k
\end{align*}
\]

with

\[
\begin{cases}
    H_c x \equiv i \left[ b(G_e(b, x) + G_D \phi_D) \right]; b \in P_N(X) \\
    \dot{G}_e = -y y^* \gamma_e; \gamma_e > 0 \\
    \dot{G}_D = -y \phi_D^* \gamma_D; \gamma_D > 0
\end{cases}
\]

Marginally Stable
Adaptive Control: Convergence to a Decoherence-Free Subspace 1

Let $S$ be an $N$-dimensional $A$-invariant subspace $P_N$ orthogonal projection onto $S$ & $P_R = I - P_N$ orthogonal projection onto $S^\perp$

\[
\begin{align*}
\frac{\partial P_N x}{\partial t} &= P_N \frac{\partial x}{\partial t} = (P_N A P_N) P_N x \\
\frac{\partial P_R x}{\partial t} &= P_R \frac{\partial x}{\partial t} = (P_R A P_R) P_R x + (P_R B) u \\
y &= (CP_R) P_R x
\end{align*}
\]

\[
\begin{align*}
\frac{\partial x_N}{\partial t} &= A_N x_N \\
\frac{\partial x_R}{\partial t} &= A_R x_R + B_R u \\
y &= C_R x_R
\end{align*}
\]
Adaptive Control: Convergence to a Decoherence-Free Subspace 2

Choosing actuators $b_i$ & sensors $c_j \Rightarrow B_N = P_N B = 0 \& C_N = P_N C = 0$

\[
\begin{align*}
\frac{\partial P_N x}{\partial t} &= P_N \frac{\partial x}{\partial t} = (P_N AP_N)_{A_N} P_N x + (P_N AP_R)_{A_{NR} = 0} P_R x + (P_N B)_{B_N} u \\
\frac{\partial P_R x}{\partial t} &= P_R \frac{\partial x}{\partial t} = (P_R AP_N)_{A_{RN} = 0} P_N x + (P_R AP_R)_{A_R} P_R x + (P_R B)_{B_R} u \\
y &= (CP_N)_{C_N} P_N x + (CP_R)_{C_R} P_R x
\end{align*}
\]
Adaptive Control: Convergence to a Decoherence-Free Subspace 3

Theorem: If \((A_R, B_R, C_R)\) is ASD, i.e. \(C_R B_R > 0 \& C_R (sI - A_R) B_R\) minimum phase, then the Adaptive Controller:

\[
\begin{align*}
    u &= G_e y + G_D \varphi_D \\
    \dot{G}_e &= -y y^* \gamma_e \\
    \dot{G}_D &= -y \varphi_D^* \gamma_D
\end{align*}
\]

will produce \(\|x_R\| = \|P_R x\|_{t \to \infty} \to 0\) (Convergence of the state \(x\) to the subspace \(S\))

Choose \(S\) as a "Decoherence-Free Subspace" (see references [20]-[22]) which are finite dimensional Hamiltonian -Invariant (A-Invariant) subspaces of the Schrodinger PDE where all decoherence effects are removed, i.e. the Schrodinger Evolution Operator is unitary.
Direct Adaptive Control of Infinite Dimensional Linear Systems

- Can be used on a Quantum System to cause it to converge to a Decoherence-Free subspace
- This requires careful Selection of actuators and Sensors
- So decoherence in Quantum Computing Gates can potentially be reduced by direct adaptive control
- Implementation in simple quantum systems is not trivial and certainly remains to be developed.
“No intelligent idea can gain general acceptance unless some stupidity is mixed in with it” .....  
Fernando Pessoa, The Book of Disquiet