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Instituto de Desenvolvimento de Novas Tecnologias

FRACTIONAL SIGNALS AND SYSTEMS

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Contents

- *Fractional? Where? (some examples)*
- *The causal fractional derivatives*
- *The fractional linear system concept*
  - *The transfer function/frequency response*
  - *The impulse response*
  - *Examples*
- *Stability*
- *Initial conditions*

Existence of fractional order systems

- Wheather/climate
- Economy/finance
- Biology/Genetics
- Music
- Biomedics
- Physics
- …

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Fractionality in Nature and Science

- ***1/f noises***
- ***Long range processes (Economy, Hydrology)***
- ***The fractional Brownian motion***
- ***The constant phase elements***
- ***Music spectrum***
- ***Network traffic***
- ***Biological processes - Deterministic Genetic Oscillation***
- ***Heat Conduction in a Porous Medium***
- ***Geometry***

Rule of thumb

- Self-similar
- Scale-free/Scale-invariant
- Power law
- Long range dependence (LRD)
- $1/f^{a}$ noise

- Porous media
- Granular
- Lossy
- Anomaly
- Disorder
- Soil, tissue, electrodes, bio, nano, network, transport, diffusion, soft matters …

Engineering applications

Control
Filtering
Image processing
System modelling – NMR, Diffusion, respiratory system, muscles, neurons
Calculus of variations - Optimization
Chaos
Fractals

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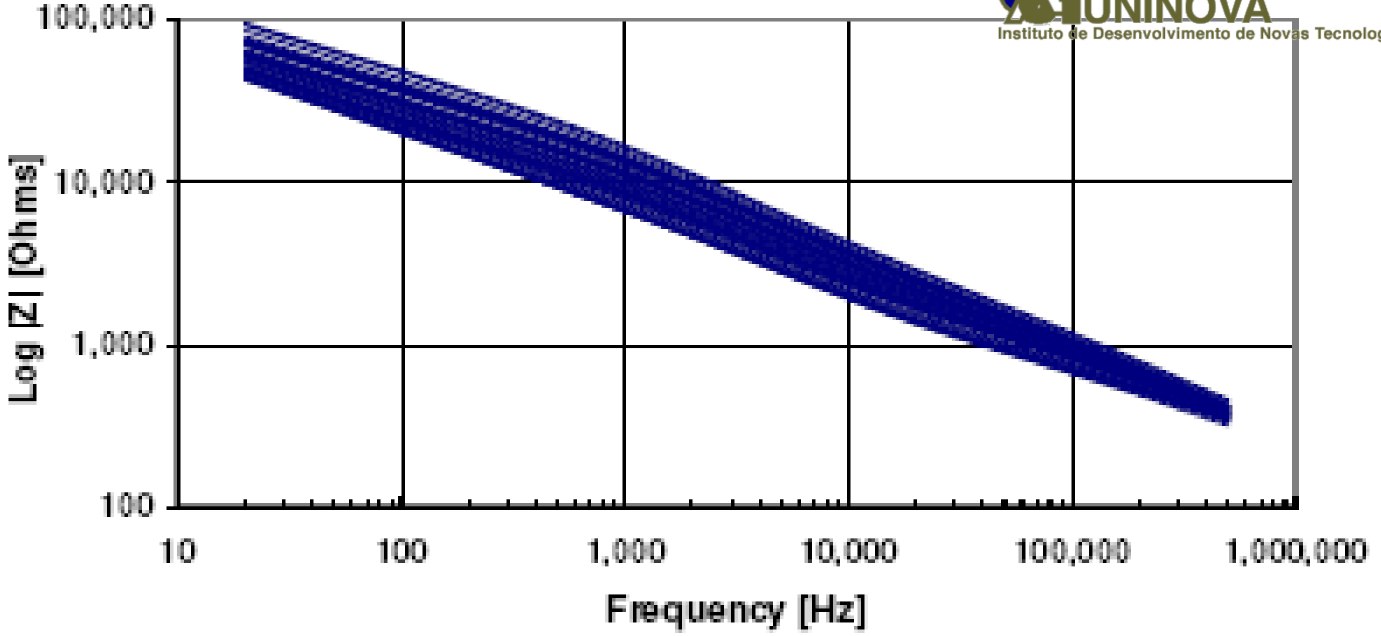
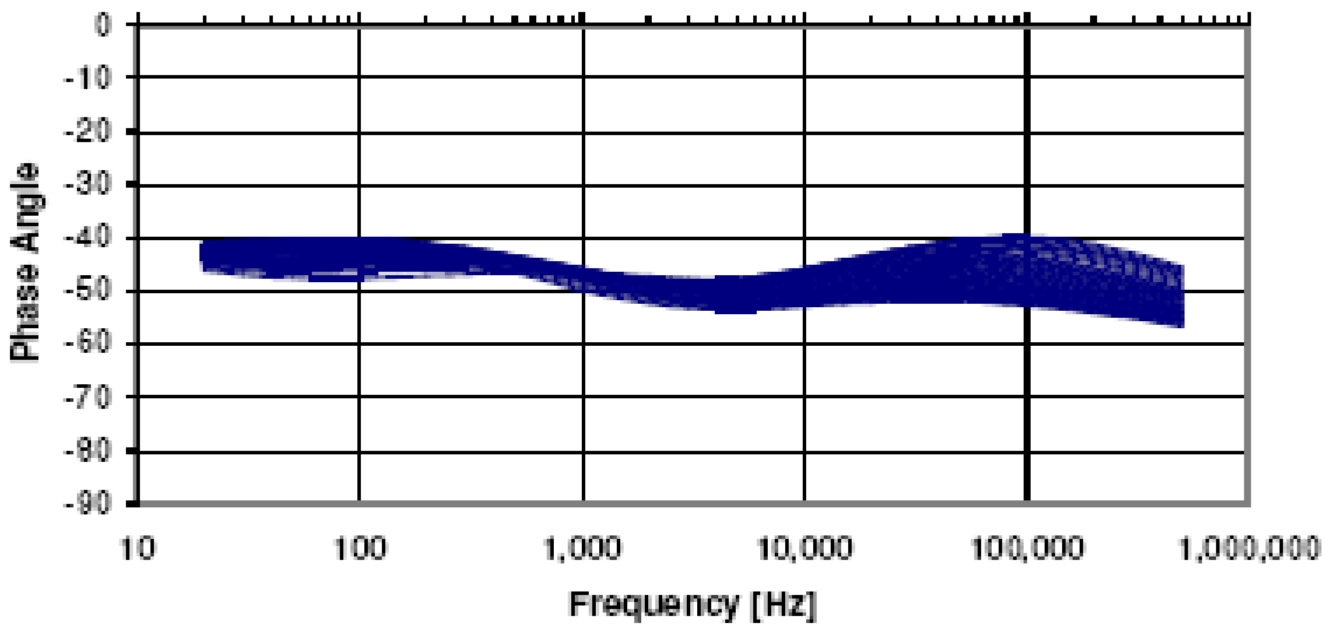
Example: The fractor

$$Z(s) = \frac{K}{s^{\alpha}}$$

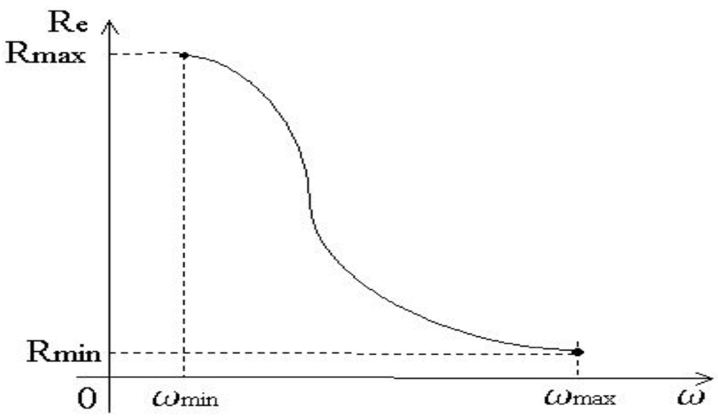


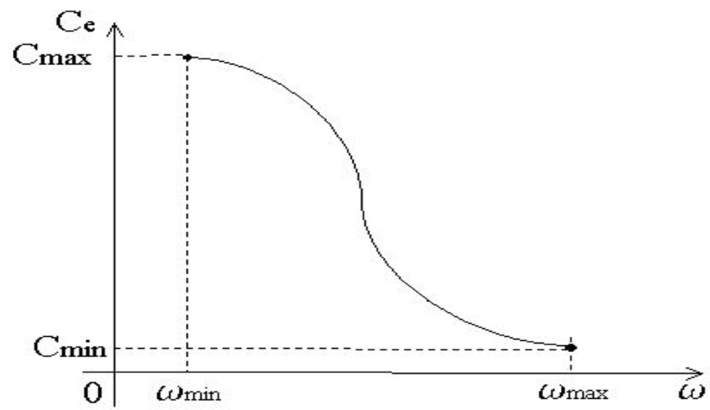
(a)

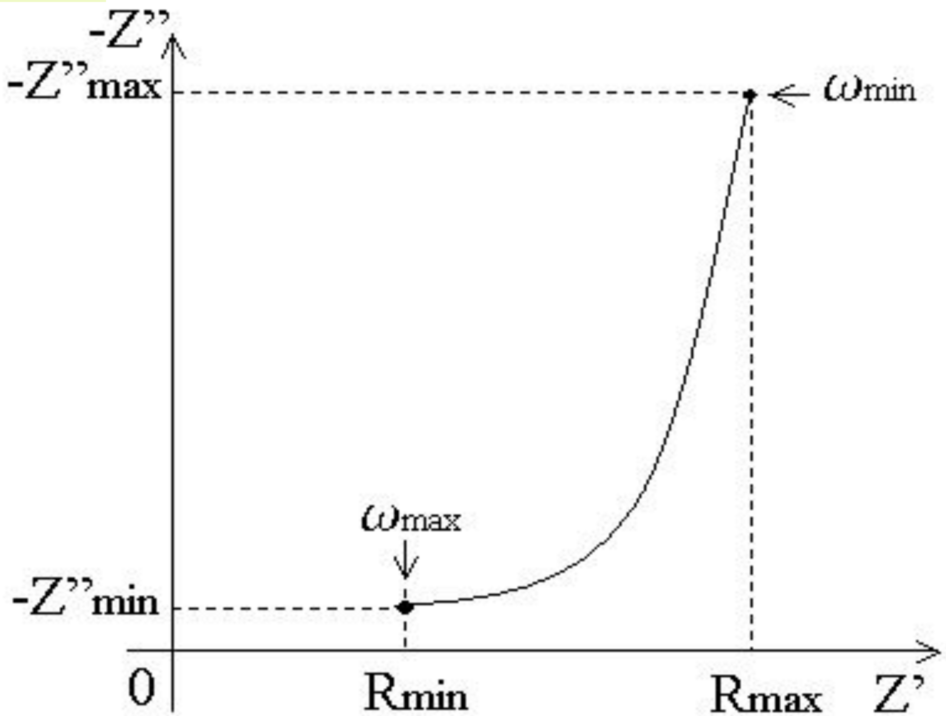


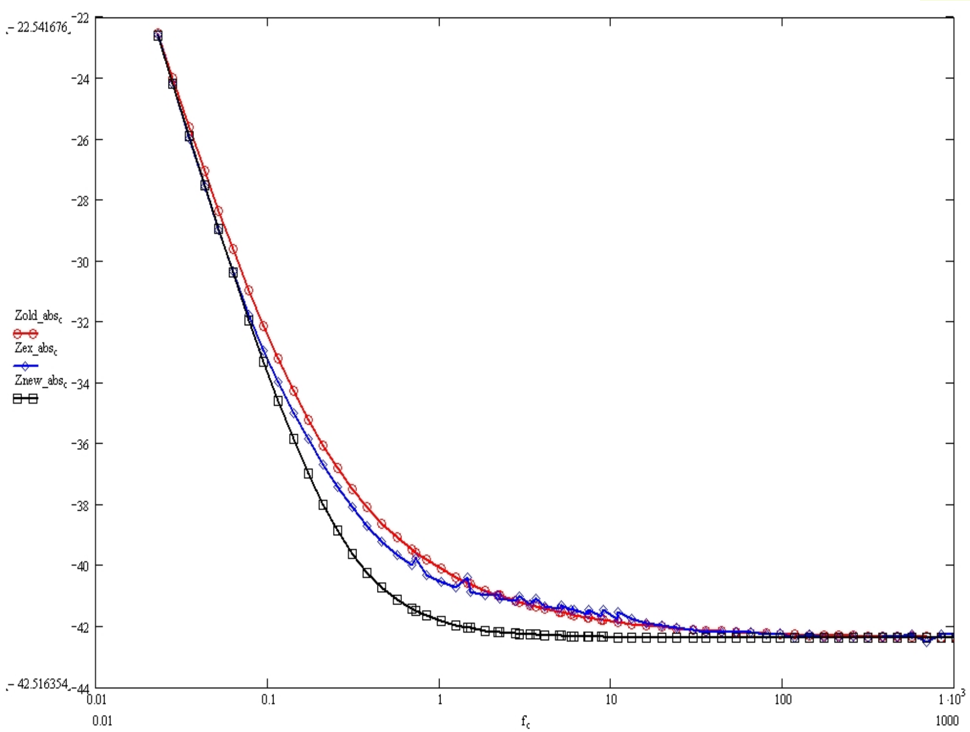
(b)

Example: supercapacitor



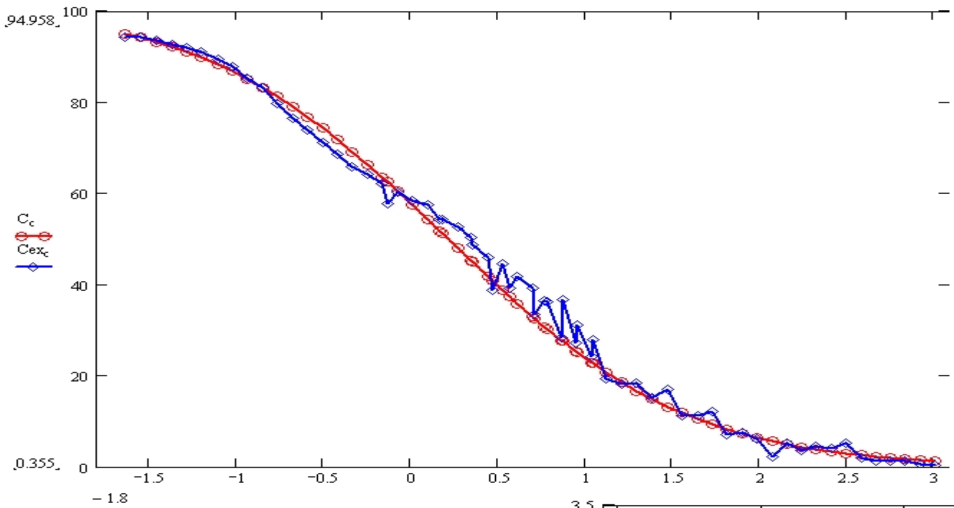


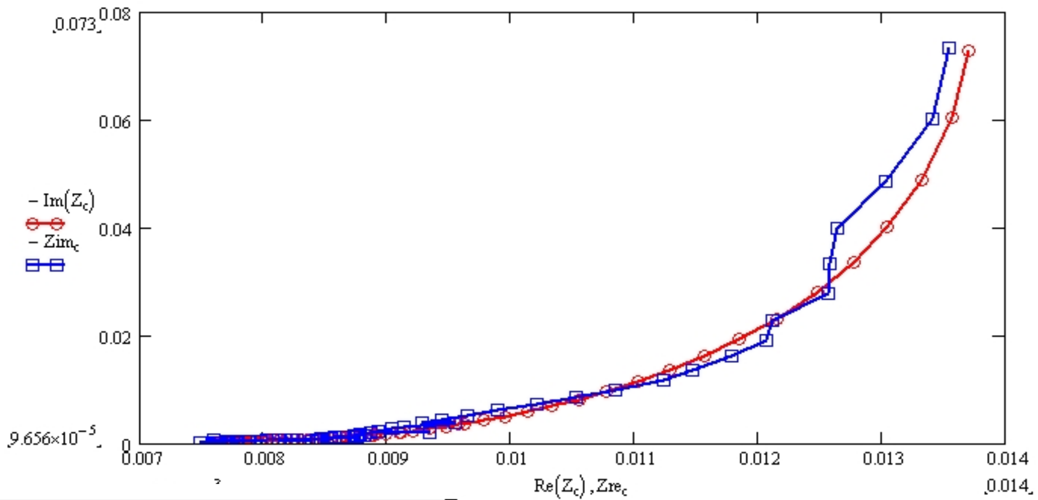


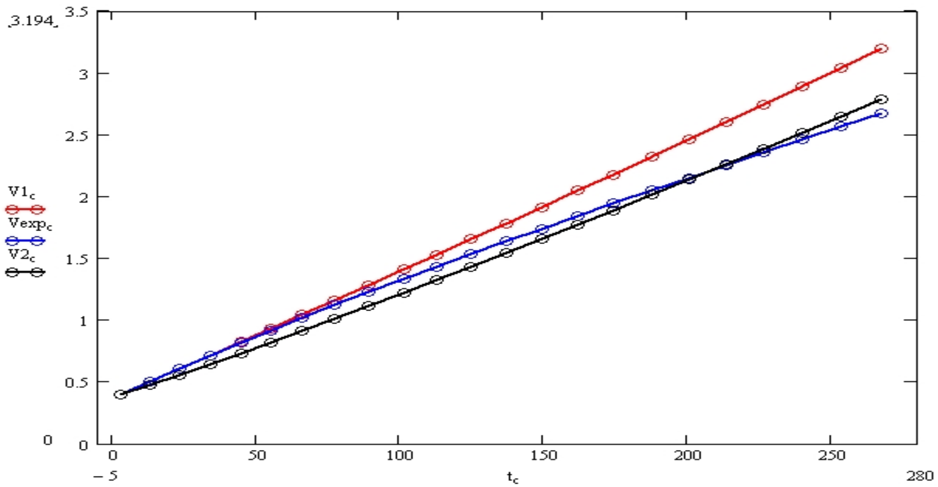


Example: supercapacitor

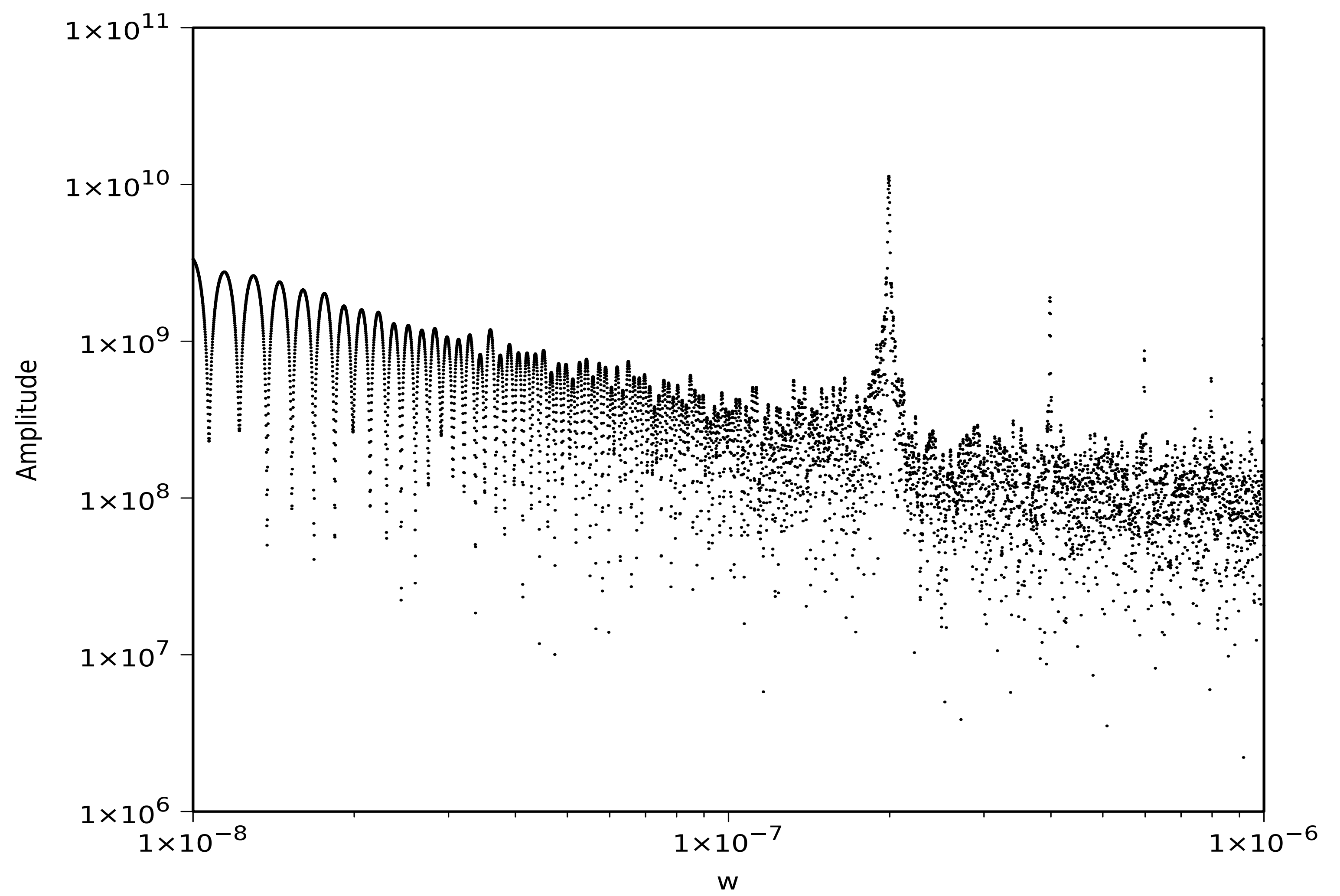
$$H(s) = 7.39 \cdot 10^{-3} + \frac{3.24 \cdot 10^{-3}}{s^{b}} + \frac{7.68 \cdot 10^{-3}}{s^{a}} + \frac{3.37 \cdot 10^{-3}}{s^{a+b}}$$





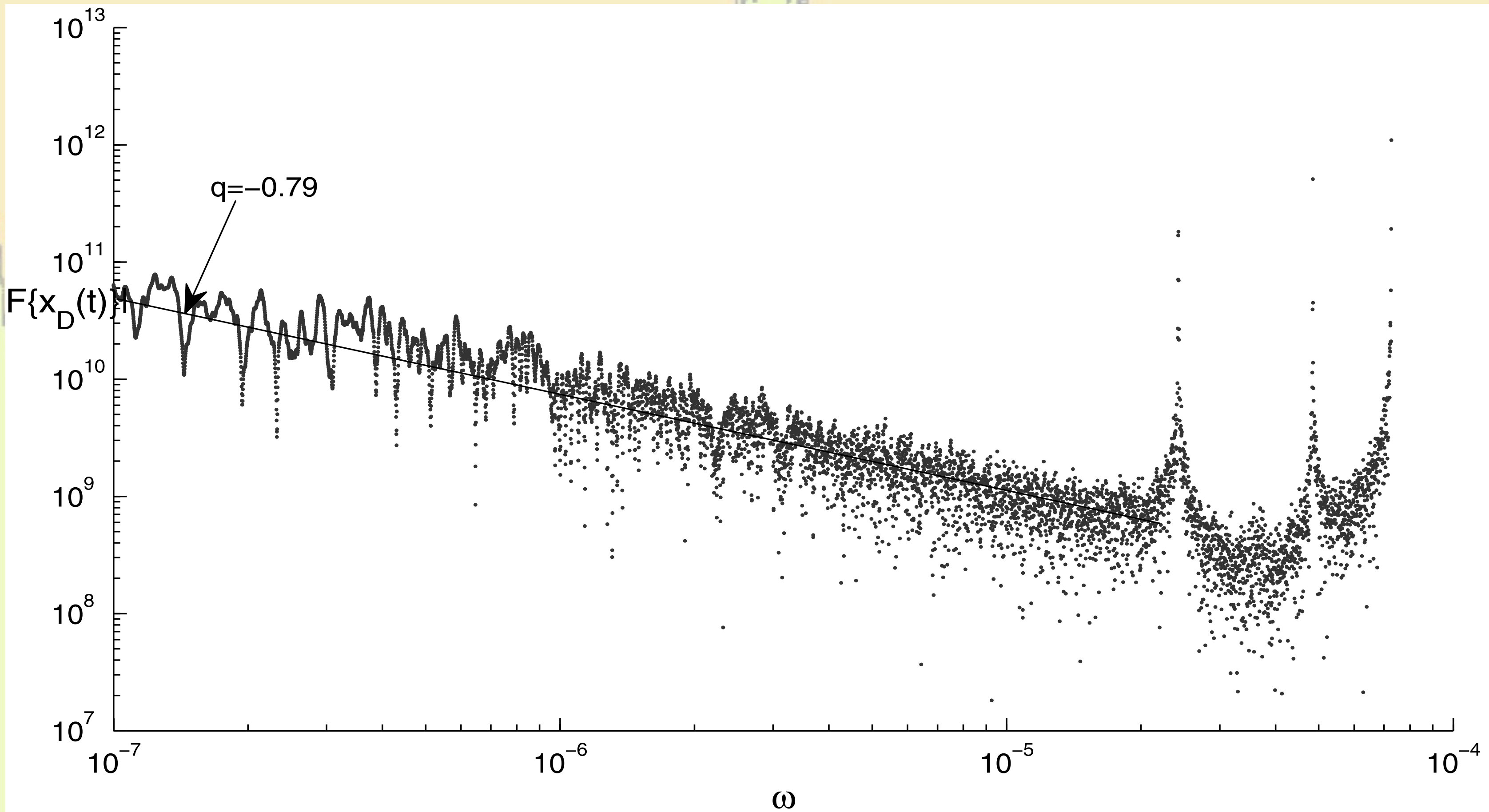


Spectrum of the monthly average temperatures of Lisbon (1881-2011)

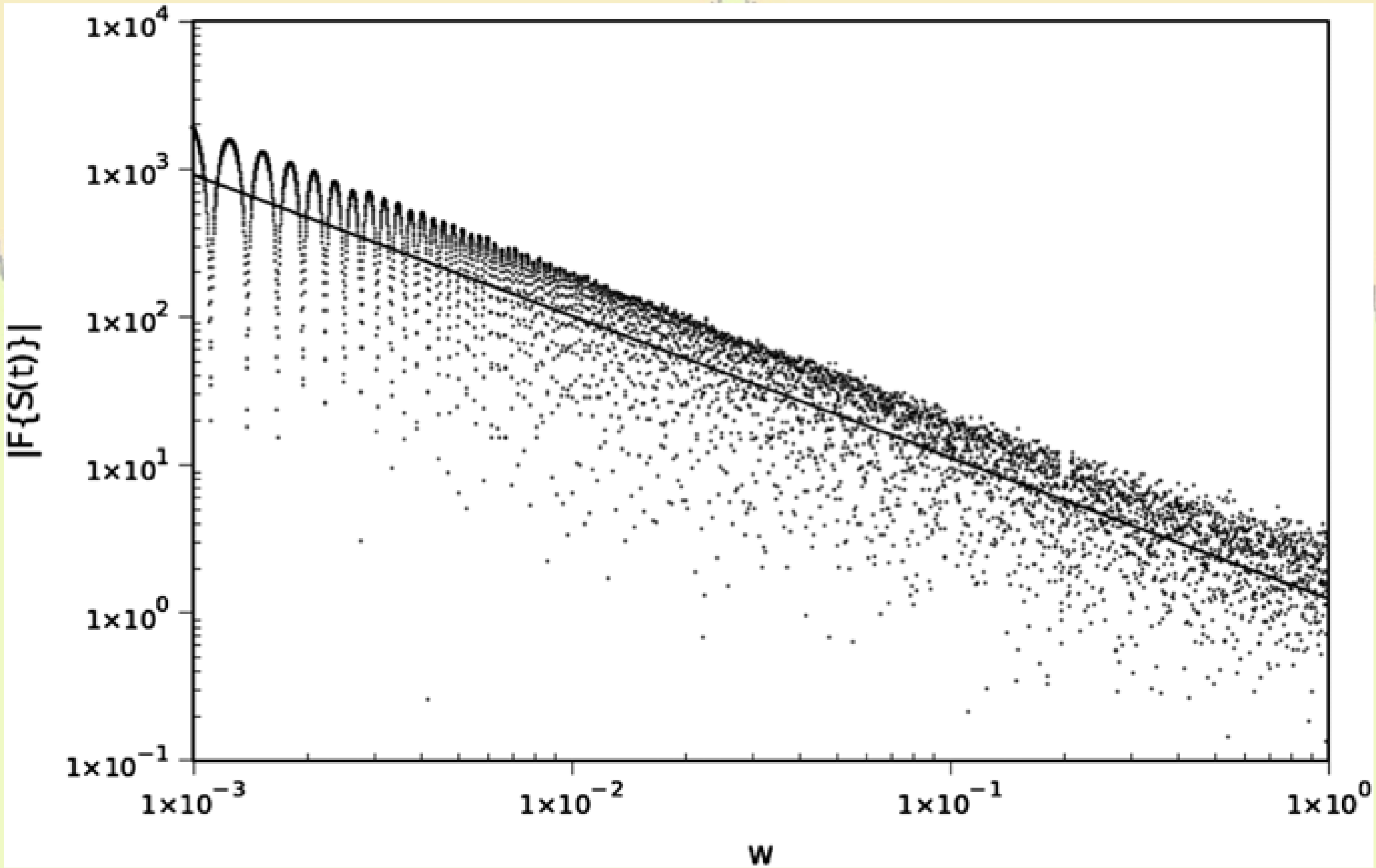


This and the next few slides were done by Prof. Tenreiro Machado: "And I say to myself: "What a fractional world", FCAA, Vol.14, No 4, 2011

Dow Jones average index (FT)



Fourier transform of the signal for the Human chromosome 1



Impedance of vegetables

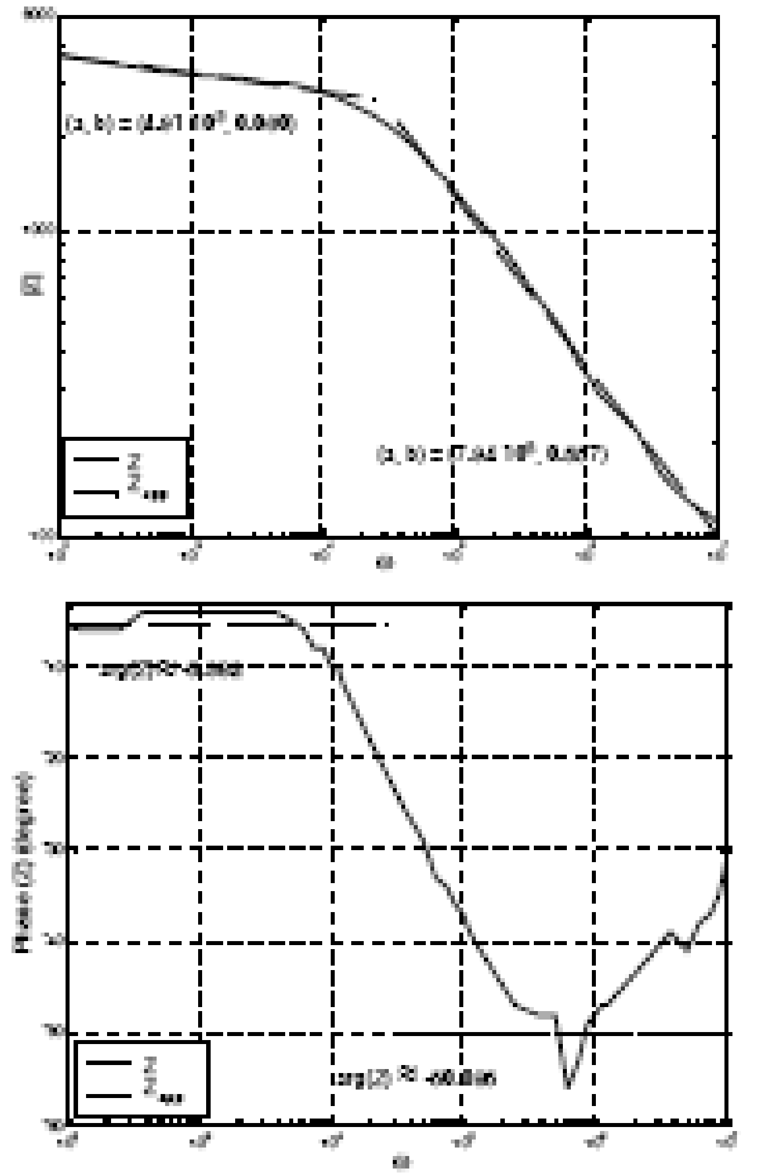
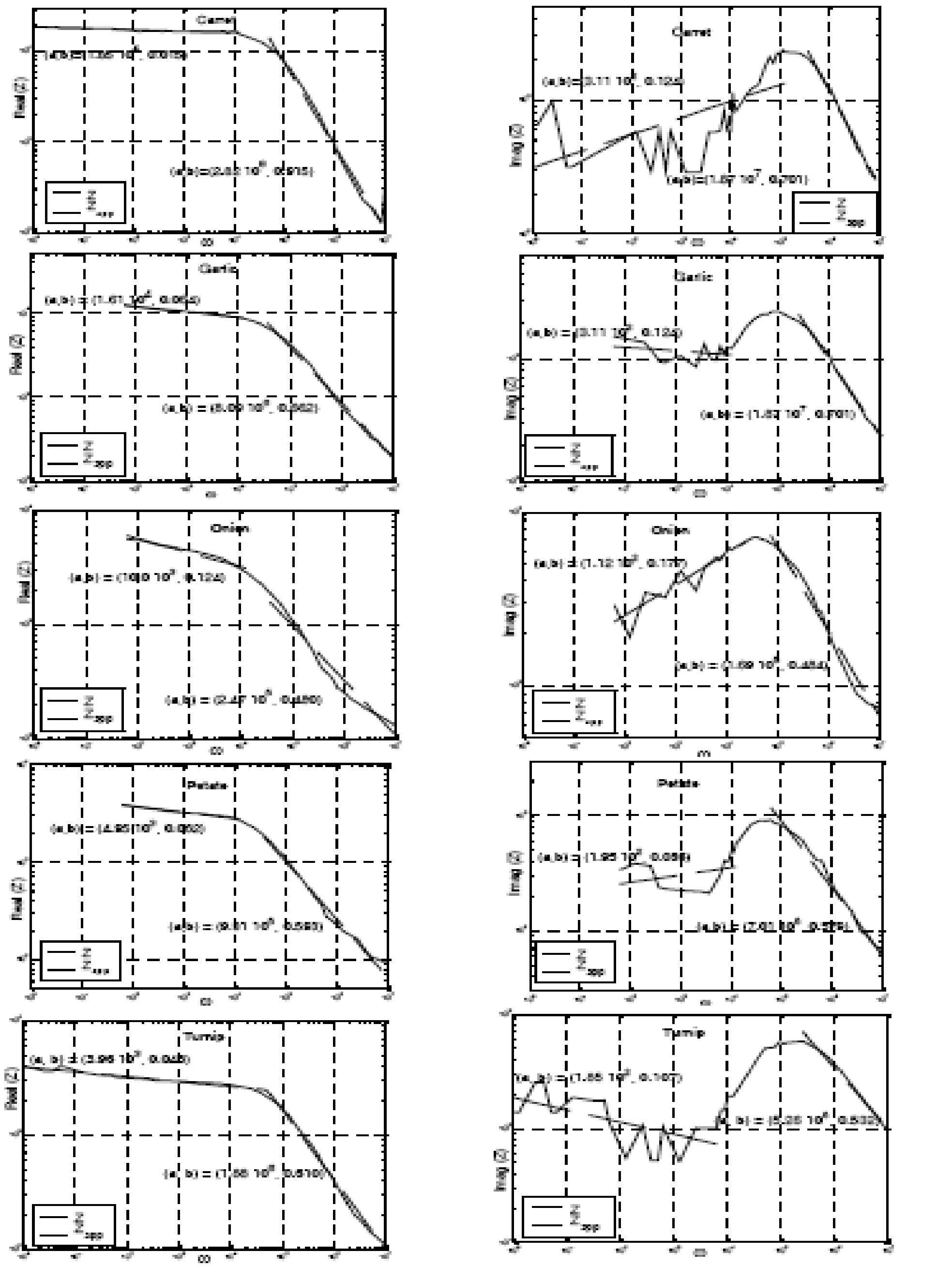
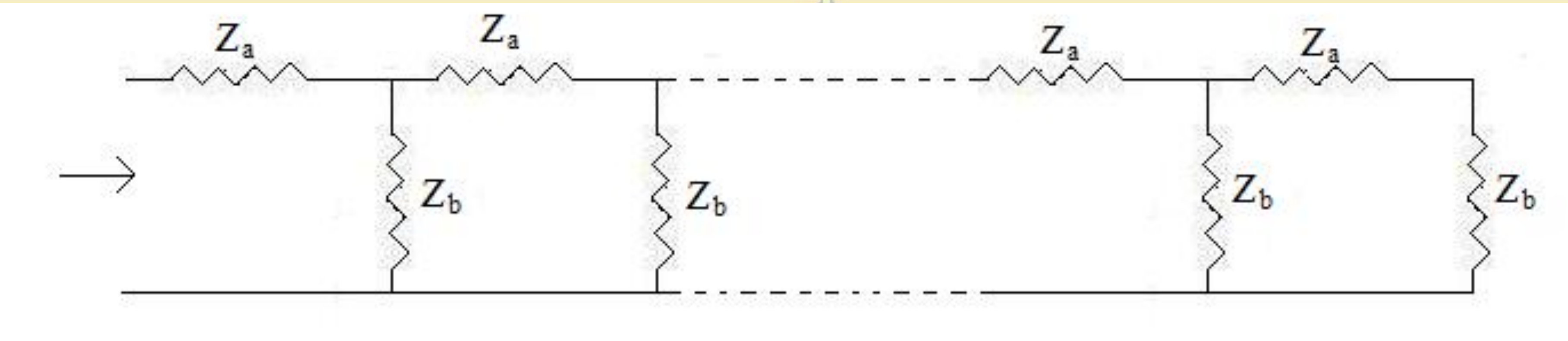


Fig 2. Bode diagrams of the impedance $Z(j\omega)$ for the potato.



Infinite Transmission line

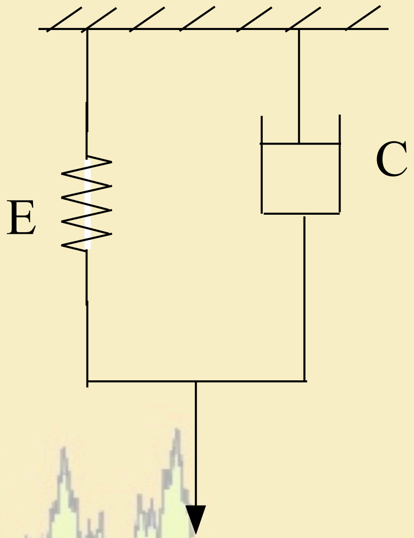


Equivalent impedance $Z = \sqrt{Z_a Z_b}$

when $Z_a = R$ and $Z_b = \frac{1}{SC}$

$$Z = \sqrt{\frac{R}{C}} S^{-1/2}$$ (Fractional order system)

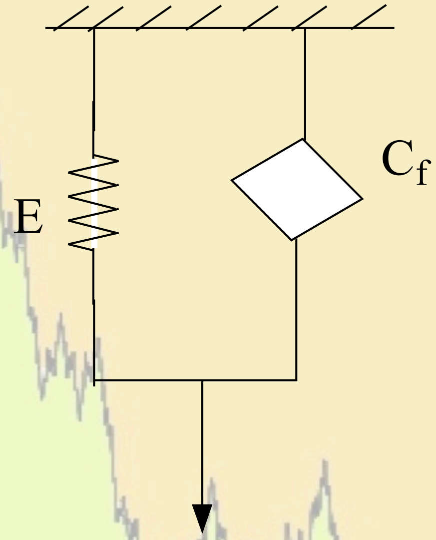
Viscoelasticity



Kelvin-Voigt model

$$\sigma(t) = \left[E + C \frac{d}{dt} \right] \varepsilon(t)$$

Integer order model



Fractional Kelvin-Voigt model

$$\sigma(t) = \left[E + C_f \frac{d^\alpha}{dt^\alpha} \right] \varepsilon(t)$$

Fractional order model

fBm – conventional formulation

$$B_H(t)-_H(0)=\frac{1}{\Gamma(H+1/2)}\left\{\int_{-\infty}^{0}\left[(t-\tau)^{H-1/2}-(-\tau)^{H-1/2}\right]w(\tau)d\tau\right\}+$$

$$+\frac{1}{\Gamma(H+1/2)}\left\{+\int_{0}^{t}(t-\tau)^{H-1/2}\,w(\tau)d\tau\right\}$$

fBm – general case

$$B_H(t)-B_H(0) = \int_0^t D^{\alpha} w(\tau)\, d\tau$$

The Laplace Transform(s)

$$\text{One-sided LT:} \Rightarrow F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$LT[f^{(\alpha)}(t))] = s^{\alpha} F(s) \; - \sum_{i=0}^{n-1} [D^{\alpha-1-i} f(0^{+})].s^{i}$$

The Laplace Transform(s)

Two-sided LT: $\Rightarrow F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$

$$LT[f^{(\alpha)}(t)] = s^{\alpha} F(s)$$

Fractional derivatives

Liouville **Riemann-Liouville** **Caputo**

Riesz **Weyl**

Hadamard **Grünwald-Letnikov**

Marchaud ...

and some other with only the name.

Fractional Integral

	Definition
Liouville integral $\alpha>0$	$D^{-\alpha}\varphi(t)=\frac{1}{(-1)^{\alpha}\Gamma(\alpha)}\int_{0}^{+\infty}\varphi(t+\tau)\tau^{\alpha-1}d\tau$
Riemann integral $\alpha>0$	$D^{-\alpha}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}d\tau$
Hadamard integral	$D^{-\alpha}(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}\int_{0}^{1}\varphi(t\tau).(1-\tau)^{\alpha-1}d\tau$
Riemann-Liouville integral	$D^{-\alpha}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}\frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}d\tau \quad \alpha>0$
Backward Riemann-Liouville integral	$D^{-\alpha}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_{t}^{b}\frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}d\tau \quad \alpha>0$
Generalised function (Cauchy)	$D^{-\alpha}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}\varphi(\tau).(t-\tau)^{\alpha-1}d\tau$

Fractional Derivative

	Definition
Left side Riemann-Liouville derivative	$D^{\alpha}\varphi(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \varphi(\tau).(t-\tau)^{\alpha-n-1}d\tau \quad t>a$
Right side Riemann-Liouville derivative	$D^{\alpha}\varphi(t)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b \varphi(\tau).(\tau-t)^{\alpha-n-1}d\tau \quad t<b$
Left side Caputo derivative	$D^{\alpha}\varphi(t)=\frac{1}{\Gamma(-\nu)}\left[\int_0^t \varphi^{(n)}(\tau).(t-\tau)^{\nu-1}\,d\tau\right] \quad t>0$
Right side Caputo derivative	$D^{\alpha}\varphi(t)=\frac{1}{\Gamma(-\nu)}\left[\int_t^{+\infty} \varphi^{(n)}(\tau).(\tau-t)^{\nu-1}\,d\tau\right]$
Generalised function (Cauchy)	$D^{\alpha}\varphi(t)=\frac{1}{\Gamma(-\alpha)}\int_{-\infty}^t \varphi(\tau).(t-\tau)^{-\alpha-1}\,d\tau$

Going into the derivative (1)

$$f_{+}^{(1)}(t) = \lim_{h \to 0} \frac{f(t) - f(t-h)}{h}$$

$$f_{-}^{(1)}(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

$$f_{0}^{(1)}(t) = \lim_{h \to 0} \frac{f(t+h/2) - f(t-h/2)}{h}$$

ARE THEY EQUIVALENT?

Going into the derivative (2)

$$f_{+}^{(1)}(t) = \lim_{h \to 0} \frac{f(t) - f(t-h)}{h} \qquad f_{-}^{(1)}(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

$$s = \lim_{h \to 0} \frac{(1 - e^{-sh})}{h} \qquad = \lim_{h \to 0} \frac{(e^{sh} - 1)}{h}$$

What happens when |s| goes to infinite?

Going into the derivative (3)

$$f_{+}^{(2)}(t) = \lim_{h \to 0} \frac{f^{(1)}(t) - f^{(1)}(t-h)}{h} =$$

$$= \lim_{h \to 0} \frac{\lim\limits_{h \to 0} \frac{f(t) - f(t-h)}{h} - \lim\limits_{h \to 0} \frac{f(t-h) - f(t-2h)}{h}}{h}$$

$$= \lim_{h \to 0} \frac{\lim\limits_{h \to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h}}{h} = \lim_{h \to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}$$

Going into the derivative (4)

$$f_{+}^{(2)}(t) = \lim_{h \to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2} \Rightarrow s^2 = \lim_{h \to 0} \frac{(1 - e^{-sh})^2}{h^2}$$

$$f_{-}^{(2)}(t) = \lim_{h \to 0} \frac{f(t+2h) - 2f(t-h) + f(t)}{h^2} \Rightarrow s^2 = \lim_{h \to 0^+} \frac{(e^{sh} - 1)^2}{h^2}$$

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Going into the derivative (5)

$$f_{+}^{(N)}(t) = \lim_{h \to 0} \frac{\sum_{k=0}^{N} (-1)^k \binom{N}{k} f(t-kh)}{h^N} \Rightarrow s^N = \lim_{h \to 0} \frac{(1 - e^{-sh})^N}{h^N}$$

$$f_{-}^{(N)}(t) = \lim_{h \to 0} \frac{(-1)^N \sum_{k=0}^{N} (-1)^k \binom{N}{k} f(t+kh)}{h^N} \Rightarrow s^N = \lim_{h \to 0} \frac{(e^{sh} - 1)^N}{h^N}$$

The N^{th}–order derivative in ONE step

Going into the anti-derivative (1)

$$f_+(t) = \lim_{h\to 0} \frac{f_+^{(-1)}(t) - f_+^{(-1)}(t-h)}{h} \Rightarrow f_+^{(-1)}(t) = \lim_{h\to 0} \left[hf_+(t) - f_+^{(-1)}(t-h) \right]$$

$$= \lim_{h\to 0} \left[hf_+(t) + hf_+(t-h) - f_+^{(-1)}(t-2h) \right]$$

$$f_+^{(-1)}(t) = \lim_{h\to 0} h \sum_{k=0}^{\infty} f(t-kh) \Rightarrow s^{-1} = \lim_{h\to 0} \frac{h}{(1-e^{-sh})} \quad Re(s) > 0$$

$$f_-^{(-1)}(t) = \lim_{h\to 0} -h \sum_{k=0}^{\infty} f(t+kh) \Rightarrow s^{-1} = \lim_{h\to 0} \frac{h}{(e^{sh}-1)} \quad Re(s) < 0$$

Essentially the Riemann integral definition!

Going into the anti-derivative (2)

$$f_{+}^{(-2)}(t) = \lim_{h\to 0} h^2 \sum_{k=0}^{\infty} (k+1) f(t-kh) \Rightarrow s^{-2} = \lim_{h\to 0} \frac{h^2}{(1-e^{-sh})^2} \quad \text{Re}(s) > 0$$

$$f_{-}^{(-2)}(t) = \lim_{h\to 0} h^2 \sum_{k=0}^{\infty} (k+1) f(t+kh) \Rightarrow s^{-2} = \lim_{h\to 0} \frac{h^2}{(e^{sh}-1)^2} \quad \text{Re}(s) < 0$$

The repeated Riemann integral!

Joining the derivative and anti-derivative transfer functions

$$s^{\pm N} = \lim_{h \to 0} \frac{(1 - e^{-sh})^{\pm N}}{h^{\pm N}} \quad Re(s) > 0$$

$$s^{\pm N} = \lim_{h \to 0} \frac{(e^{sh} - 1)^{\pm N}}{h^{\pm N}} \quad Re(s) < 0$$

Fractionalising the transfer function

$$s^{\alpha} = \lim_{h \to 0^+} \frac{(1 - e^{-sh})^{\alpha}}{h^{\alpha}} = \lim_{h \to 0^+} \frac{(e^{sh} - 1)^{\alpha}}{h^{\alpha}}$$

Re(s) > 0 Re(s) < 0

We must be careful with the branch cut lines due to the branch point at s=0

The differintegrator

$$s^{\alpha} = \lim_{h \to 0^{+}} \frac{(1 - e^{-sh})^{\alpha}}{h^{\alpha}} = \lim_{h \to 0^{+}} \frac{(e^{sh} - 1)^{\alpha}}{h^{\alpha}}$$

Re(s) > 0

Re(s) < 0

Here it is the causality!

Generalisation of a well known property of the Laplace transform

$[D_f^{\alpha} f(t)] = s^{\alpha} F(s)$ **for Re(s) > 0 Forward**

$[D_b^{\alpha} f(t)] = s^{\alpha} F(s)$ **for Re(s) < 0 Backward**

There is a system – the differintegrator - that has s^{α} as transfer function.

Fractional Differintegrator

- Inverse LT of s^α for Real orders:
  - Causal

$$LT^{-1}[s^\alpha] = \frac{t^{-\alpha-1}}{(\alpha-1)!}u(t)$$

  - Anti-causal

$$LT^{-1}[s^\alpha] = -\frac{t^{-\alpha-1}}{(\alpha-1)!}u(-t)$$

Liouville differintegration

- Causal

$$x_f^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} x(\tau).(t-\tau)^{-\alpha-1}\, d\tau$$

Long memory!

- Anti-causal

$$x_b^{(\alpha)}(t) = -\frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} x(\tau).(t-\tau)^{-\alpha-1}\, d\tau$$

Liouville differintegration

- Causal $$x_f^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} x(t-\tau).\tau^{-\alpha-1}\, d\tau$$
- Anti-causal $$x_b^{(\alpha)}(t) = \frac{1}{(-1)^{-\alpha}\Gamma(\alpha)} \int_0^{\infty} x(t+\tau).\tau^{-\alpha-1}\, d\tau$$

Grünwald-Letnikov fractional derivative

$$s^{\alpha} = \lim_{h \to 0^+} \frac{(1 - e^{-sh})^{\alpha}}{h^{\alpha}} = \lim_{h \to 0^+} \frac{(e^{sh} - 1)^{\alpha}}{h^{\alpha}}$$

Re(s) > 0

Re(s) < 0

$$f_f^{(\alpha)}(t) = \lim_{h \to 0^+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t-kh)}{h^{\alpha}}$$

forward

backward

$$f_b^{(\alpha)}(t) = \lim_{h \to 0^+} e^{-j\alpha\pi} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t+kh)}{h^{\alpha}}$$

The law of the exponents

- $D^{\alpha}D^{\beta} f(z) = D^{\beta}D^{\alpha} f(z) = D^{\alpha+\beta} f(z)$

- $D^{\alpha}D^{-\alpha} f(z) = D^{-\alpha}D^{\alpha} f(z) = f(z)$

Derivative of the causal power

$$\delta_f^{(-\beta-1)}(t) = \frac{t^{\beta}}{\Gamma(\beta+1)}u(t)$$

$$\delta_f^{(\alpha-\beta+1)}(t) = \frac{t^{-\alpha+\beta}}{\Gamma(\beta-\alpha+1)}u(t) \Rightarrow \mathbf{D^{\alpha}\left[t^{\beta}u(t)\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}u(t)}$$

β cannot be a negative integer

Derivative of the exponential

If $f(t) = e^{st}$

$$f_f^{(\alpha)}(t) = e^{at} \lim_{h \to 0^+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} e^{-kh}}{h^{\alpha}} = e^{at} \lim_{h \to 0^+} \frac{(1 - e^{-ah})^{\alpha}}{h^{\alpha}} = s^{\alpha} \; e^{st} \quad \text{if } \operatorname{Re}(s) > 0$$

$$f_b^{(\alpha)}(t) = e^{at} \lim_{h \to 0^+} (-1)^{\alpha} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} e^{kh}}{h^{\alpha}} = e^{at} \lim_{h \to 0^+} \frac{(e^{ah} - 1)^{\alpha}}{h^{\alpha}} = s^{\alpha} \; e^{st} \quad \text{if } \operatorname{Re}(s) < 0$$

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Forward derivative of the sinusoid

$$f(t) = e^{j\omega t} \quad \omega > 0 \Rightarrow f_f^{(\alpha)}(t) = (j\omega)^{\alpha} e^{j\omega t}$$

Then

$$D^{\alpha}\cos(\omega t) =$$

$$= D^{\alpha}[e^{j\omega t} + e^{-j\omega t}]/2 = 1/2(j\omega)^{\alpha} e^{j\omega t} + 1/2(-j\omega)^{\alpha} e^{-j\omega t} =$$

$$= \omega^{\alpha}\cos(\omega t + \alpha\pi/2)$$

and, similarly

$$D^{\alpha}\sin(\omega t) = \omega^{\alpha}\sin(\omega t + \alpha\pi/2)$$

What about the backward?

Continuous-time Shift-invariant Systems

- Diferential Equation

$$\sum_{n=0}^{N} a_n D^{\nu_n} y(t) = \sum_{m=0}^{M} b_m D^{\nu_m} x(t)$$

- Transfer Function

$$H(s)= \frac{\sum_{m=0}^{M} b_m s^{\nu_m}}{\sum_{n=0}^{N} a_n s^{\nu_n}}$$

- Only two solutions:

causal and anti-causal

Fractional Continuous-time Linear Systems

- Diferential Equation

$$\sum_{n=0}^{N} a_n D^{n\nu} y(t) = \sum_{m=0}^{M} b_m D^{m\nu} x(t)$$

- Transfer Function

$$H(s)=\frac{\sum_{m=0}^{M} b_m s^{m\nu}}{\sum_{n=0}^{N} a_n s^{n\nu}}$$

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From the Transfer Function to the Impulse Response

$$H(s) = \frac{\sum_{m=0}^{M} b_m s^{m\nu}}{\sum_{n=0}^{N} a_n s^{n\nu}} = \sum_{n=1}^{N} \frac{A_n}{s^{\nu} - p_n}$$

⇓ ⇑

$$H(w) = \frac{\sum_{m=0}^{M} b_m w^{m}}{\sum_{n=0}^{N} a_n w^{n}} = \sum_{n=1}^{N} \frac{A_n}{w - p_n}$$

Partial fraction inversion

$$\frac{1}{s^{\nu} - p} = \frac{1}{s^{\nu}} \frac{1}{1 - ps^{-\nu}} = \frac{1}{s^{\nu}} \sum_{n=0}^{\infty} p^{n} s^{-\nu n} = \sum_{n=1}^{\infty} p^{n-1} s^{-\nu n}$$

$$h(t) = \sum_{n=1}^{\infty} p^{n-1} \frac{t^{n\nu-1}}{\Gamma(n\nu)} u(t)$$

Alpha exponential

Related to the Mittag-Leffler function

If $\nu = 1$, $h(t) = e^{pt}.u(t)$, the usual solution

A practical example

The "single-degree-of-freedom fractional oscillator" consists of a mass and a fractional Kelvin element and it is applied in viscoelasticity. The equation of motion is

$$\mathbf{m D^2 x(t) + c\, D^{\alpha}\, x(t) + kx(t) = f(t)}$$

where m is the mass, c the damping constant, k the stiffness, x the displacement and f the forcing function.

Let us introduce the parameters: $\omega_0 = \sqrt{k/m}$ as the undampped natural frequency of the system $\zeta = \frac{c}{2m\omega_0^{2-\alpha}}$ and α=1/2. The transfer function is

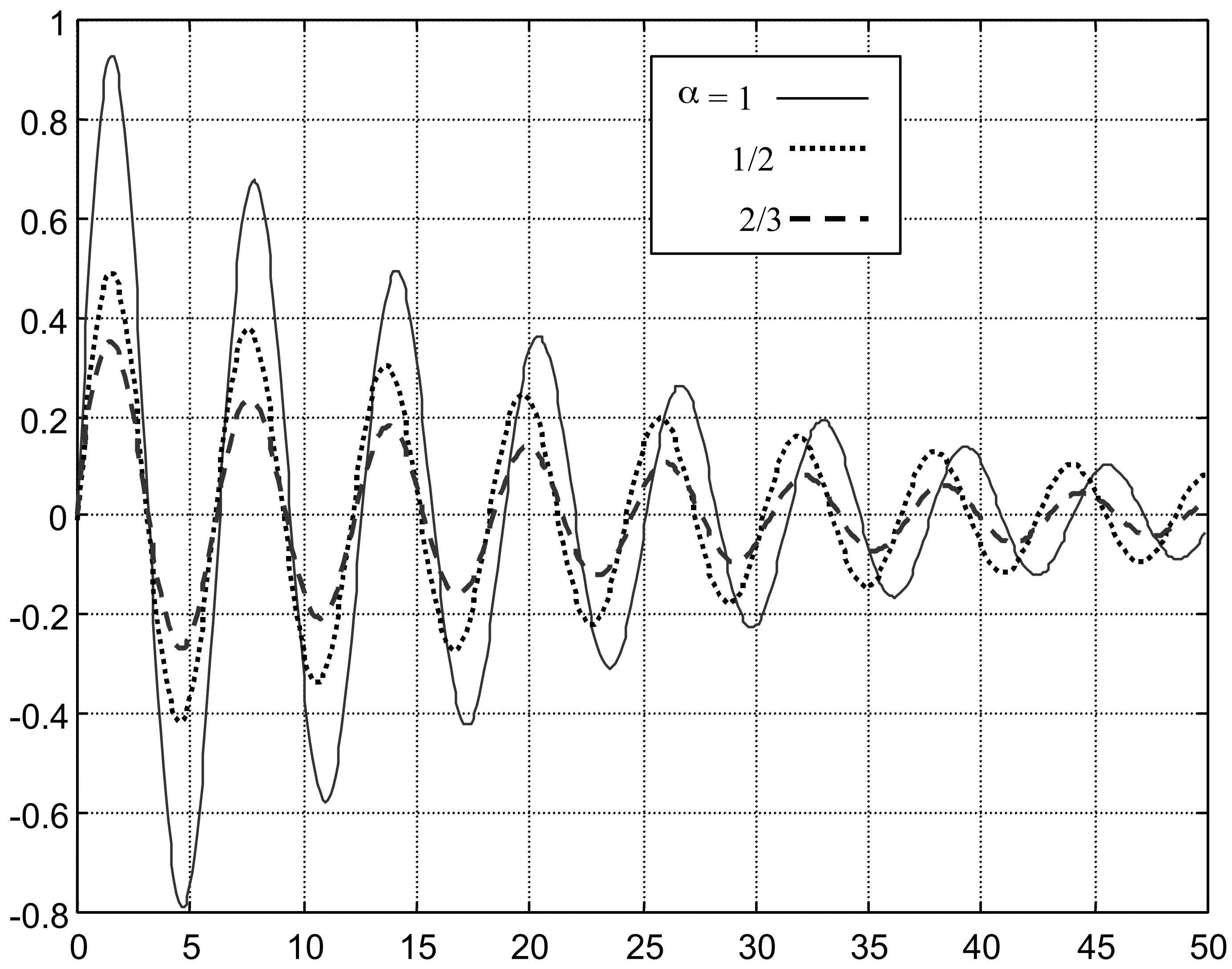
$$\mathbf{H(s) = \frac{1}{s^{\alpha} + 2\omega_0^{2-\alpha}\zeta s^{\alpha} + \omega_0^2}}$$

with indicial polynomial $s^4 + 2\omega_0^{3/2}\zeta\, s + \omega_0^2$. Its roots are on two vertical straight lines with symmetric abscissas, but only two belong to the first Riemann surface. We obtain the impulse response as:

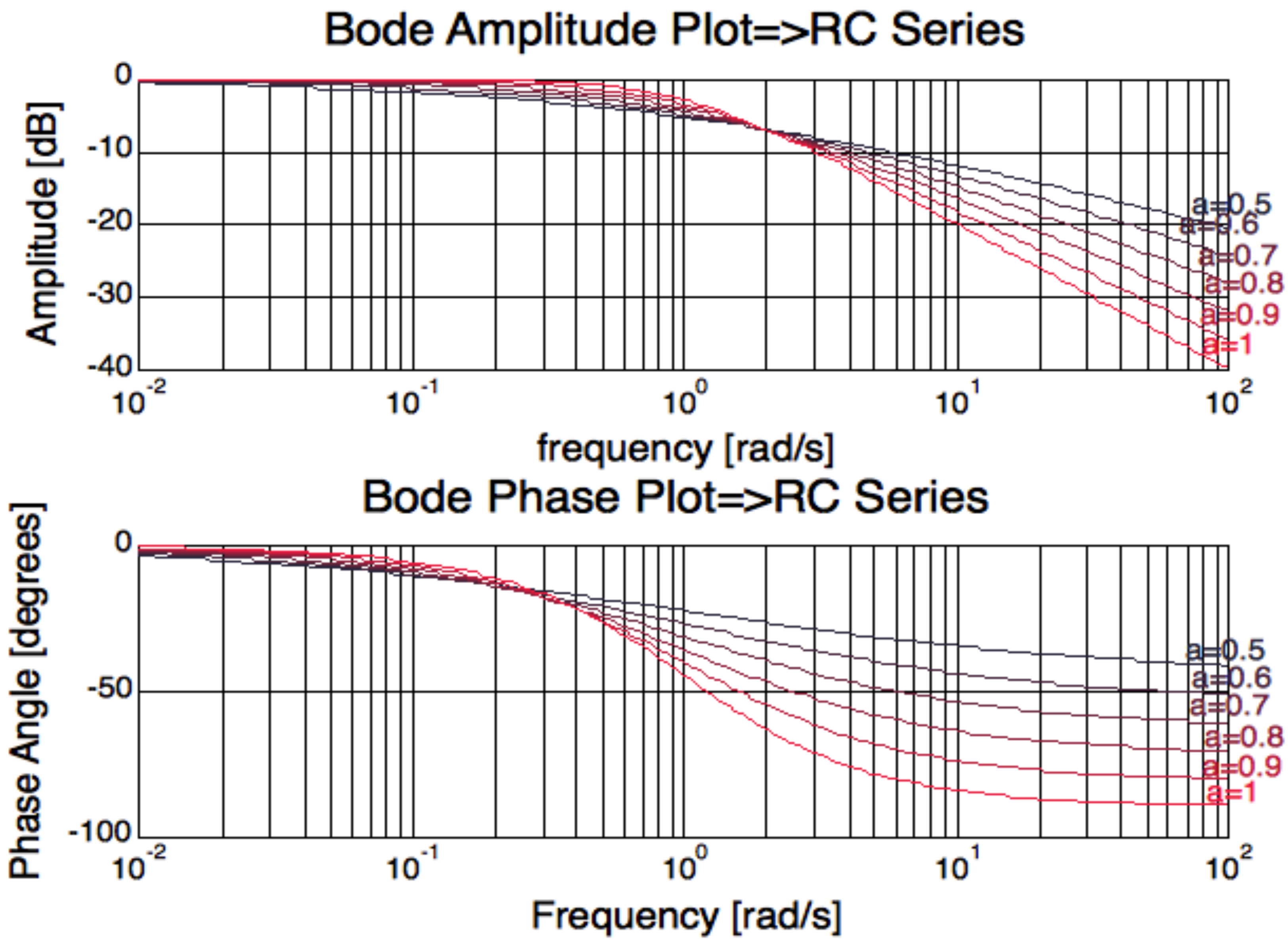
$$\mathbf{h(t) = Re\left\{ r.s_1.e^{(-0.0354 + j1.0353)t} u(t) + D^{1/2}\left[\ r.e^{(-0.0354 + j1.0353)t} u(t)\right] \right\}}$$

where r is the residue at s_1.

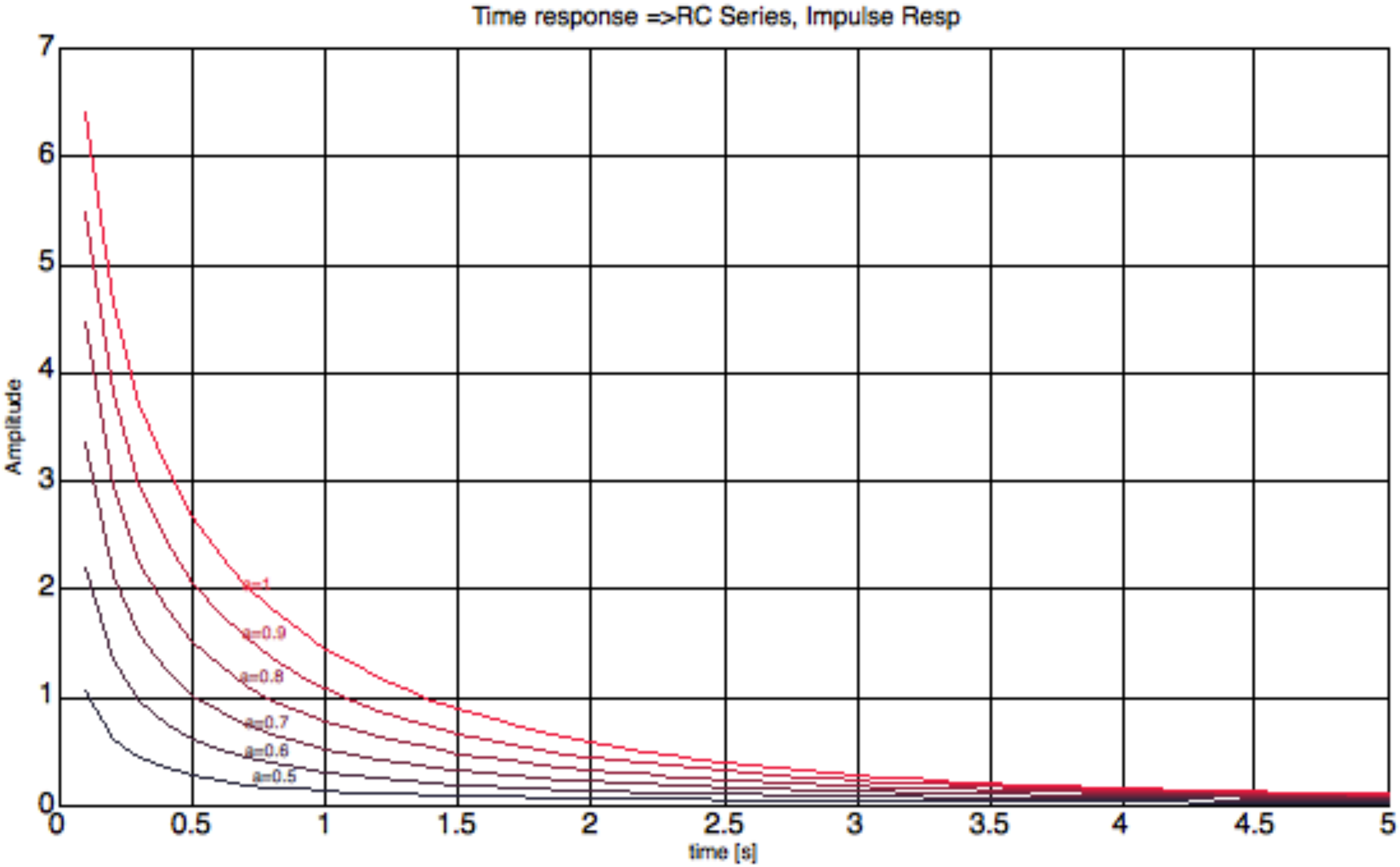
A practical example (cont.)



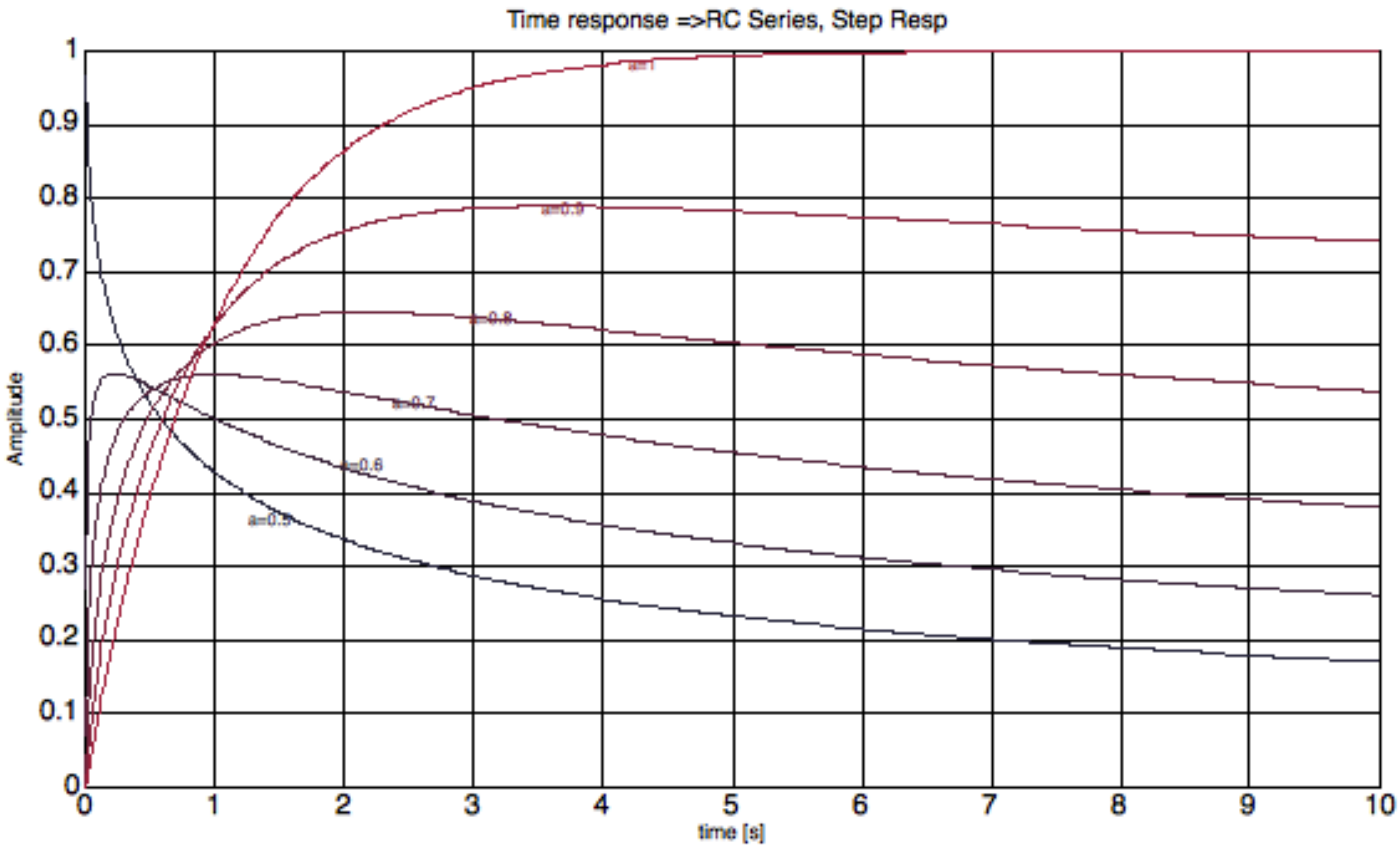
An RC circuit



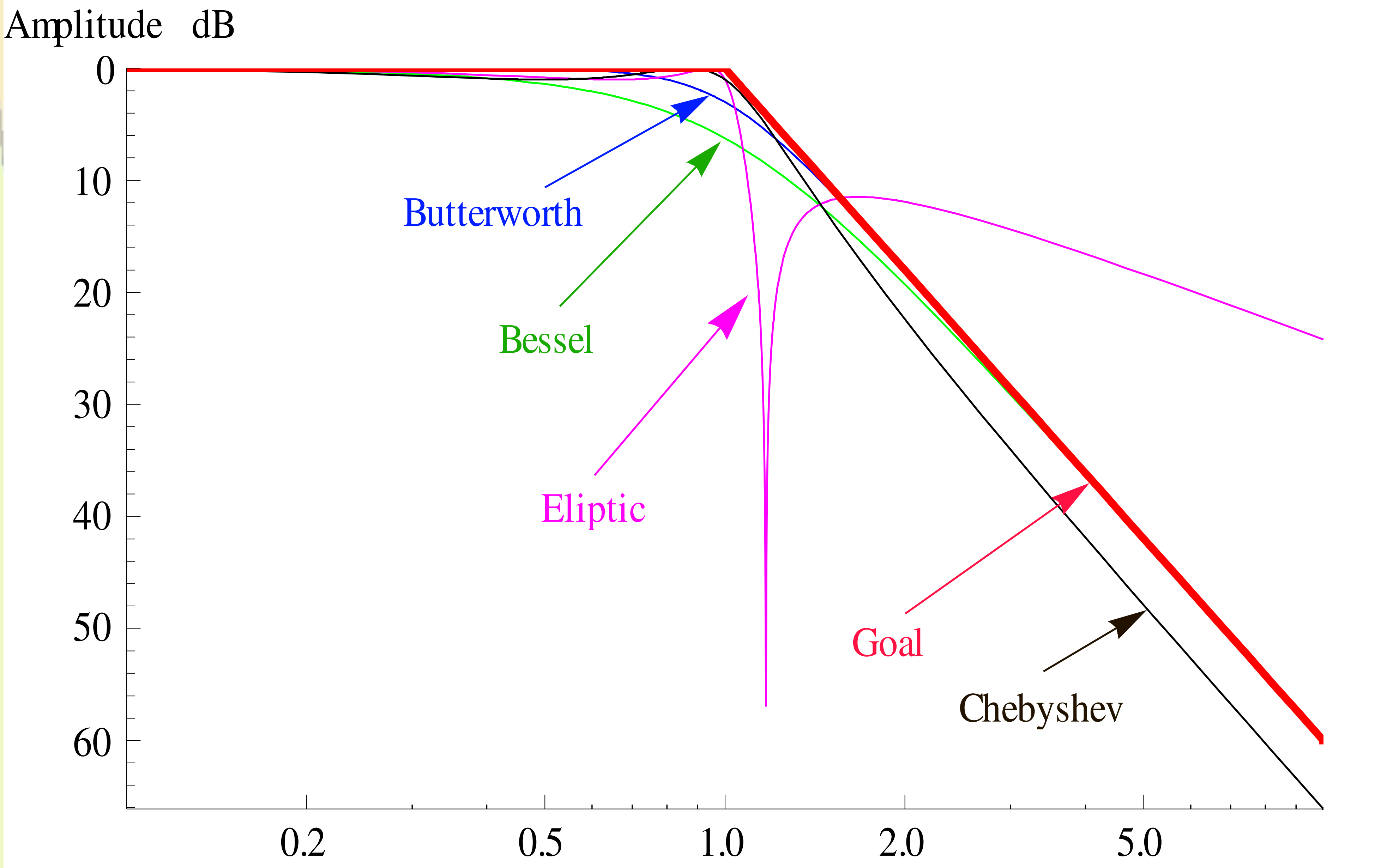
Time response RC circuit



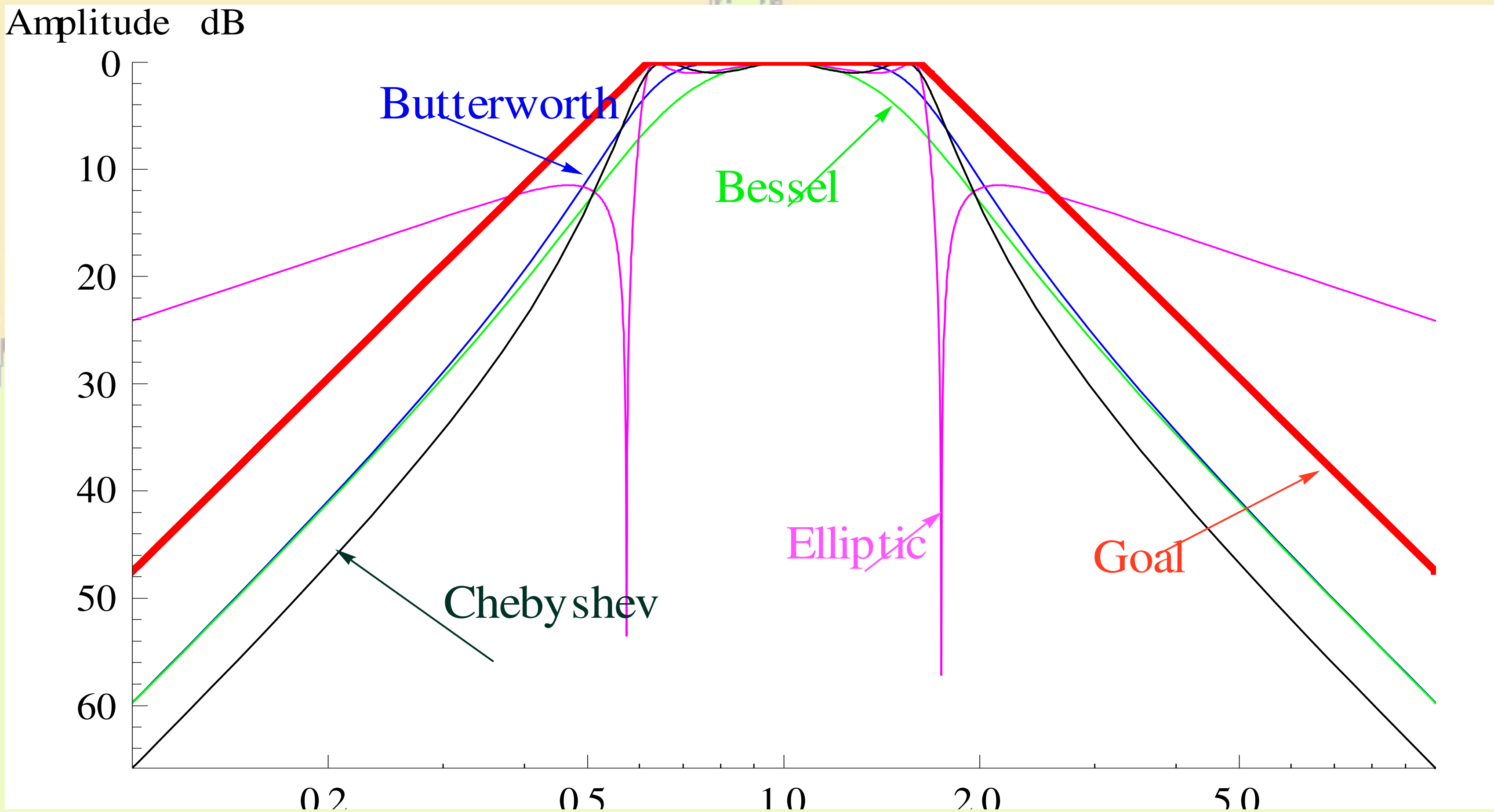
Step response of RC circuit



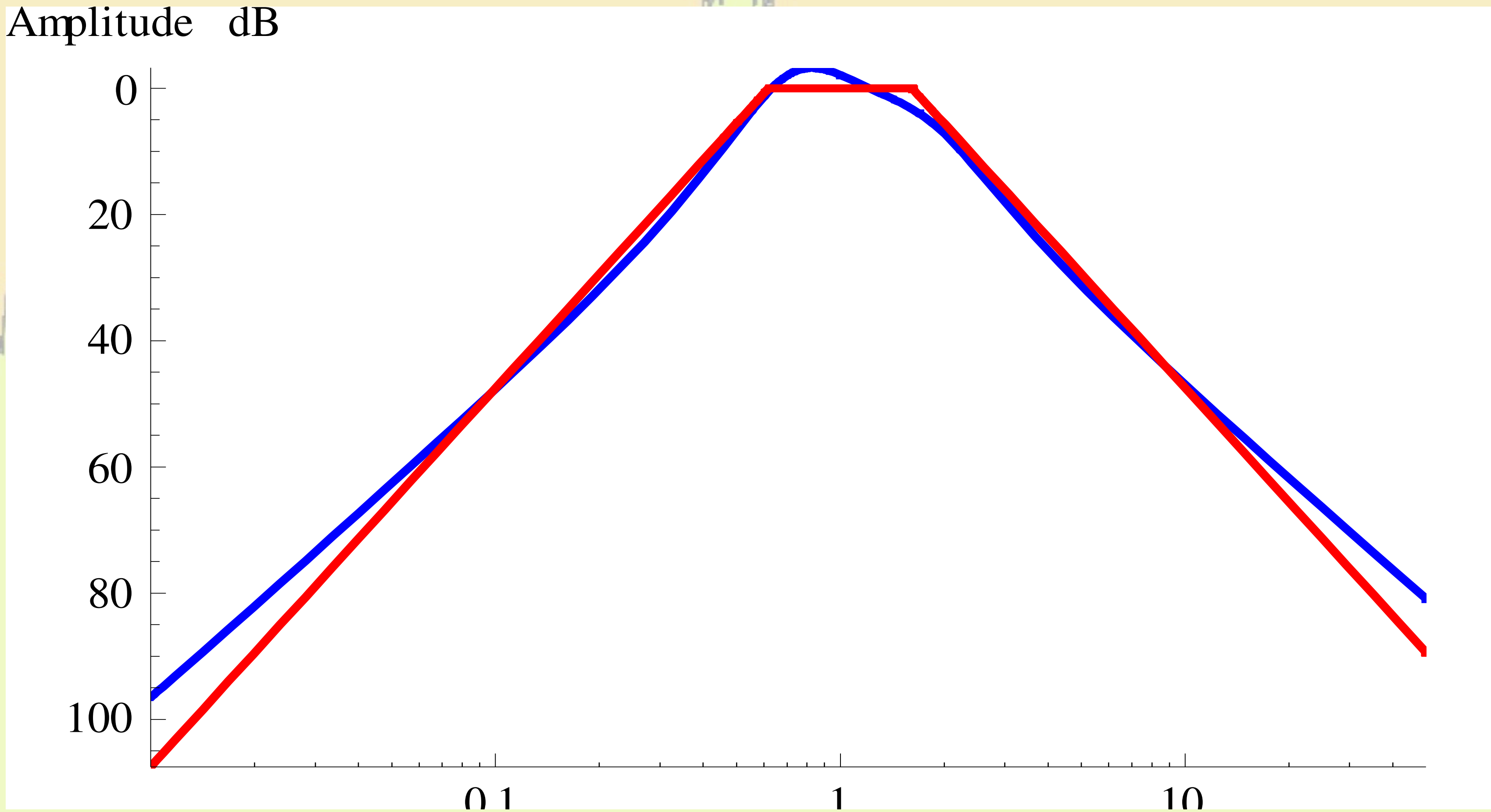
Design from classics



Design from classics

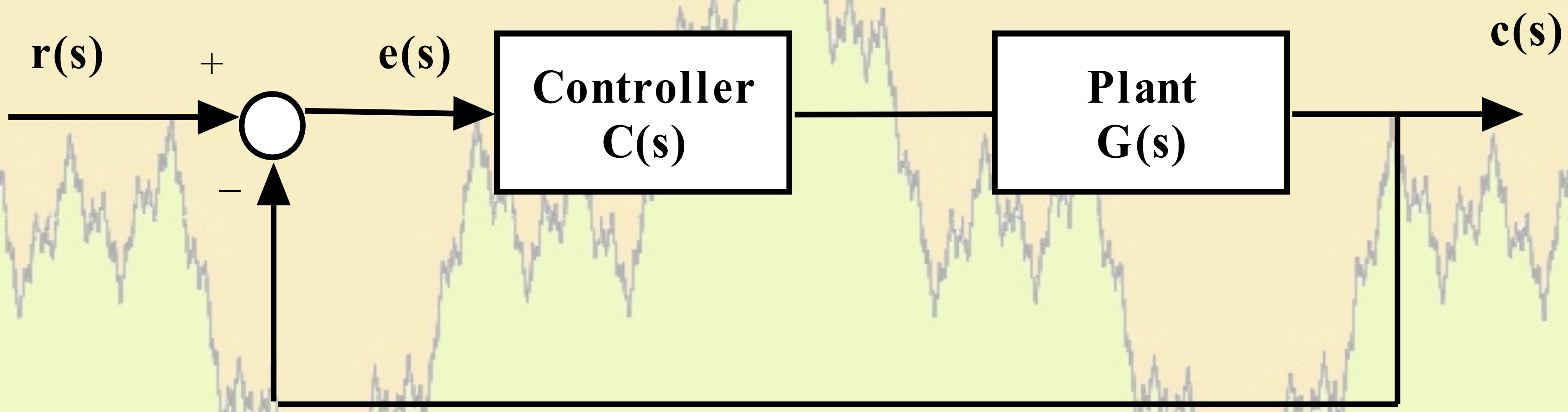


Bandpass filter



Fractional PID Controller:

$$C(s) = k_p(1 + k_D s^{\alpha} + k_I s^{-\beta})$$

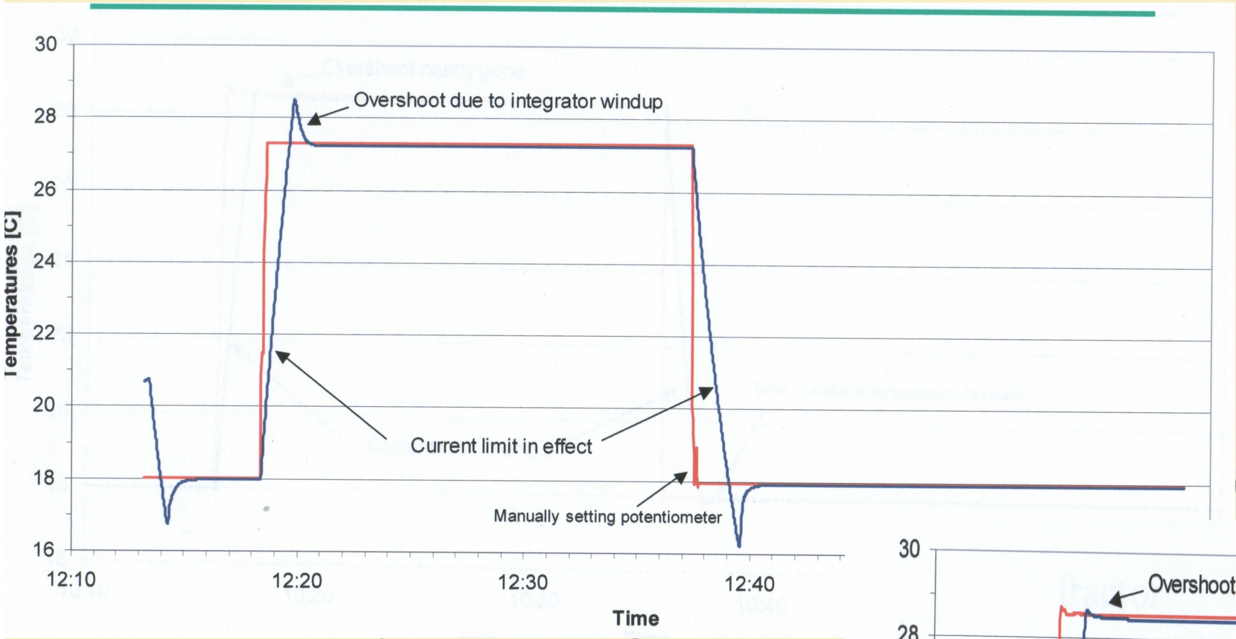


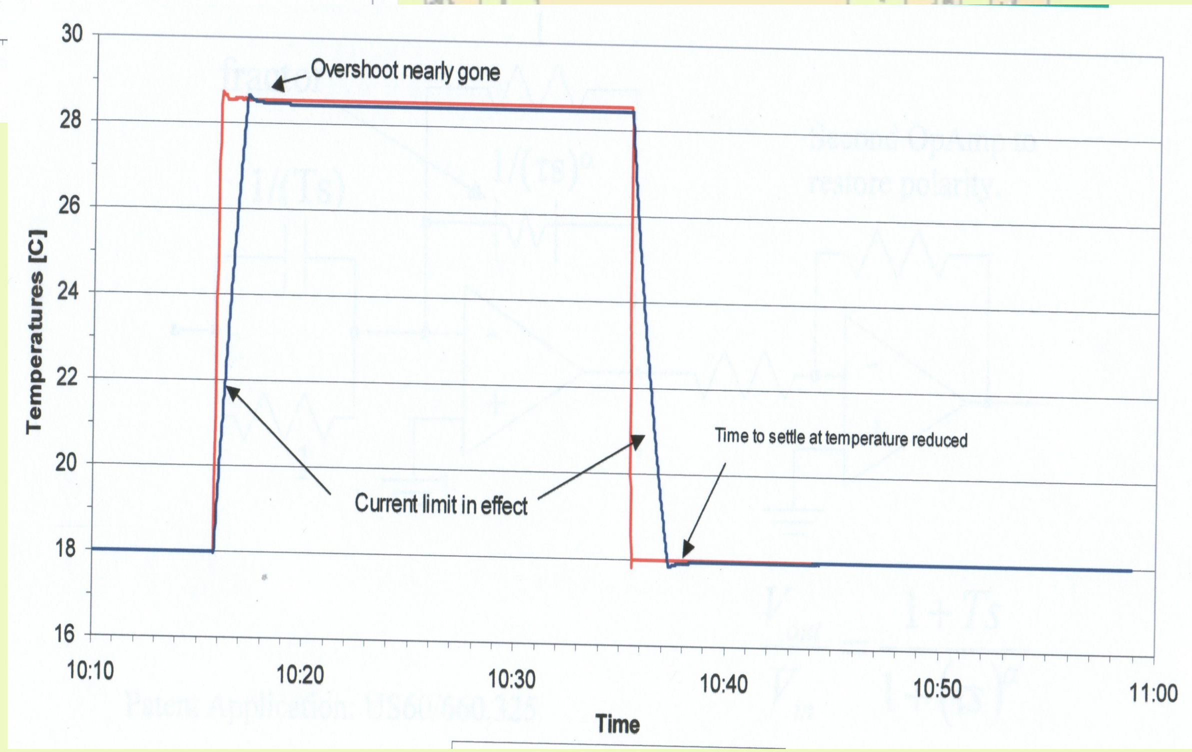
<u>Advantage:</u> More flexibility, so better performance expected.

<u>Challenges:</u>

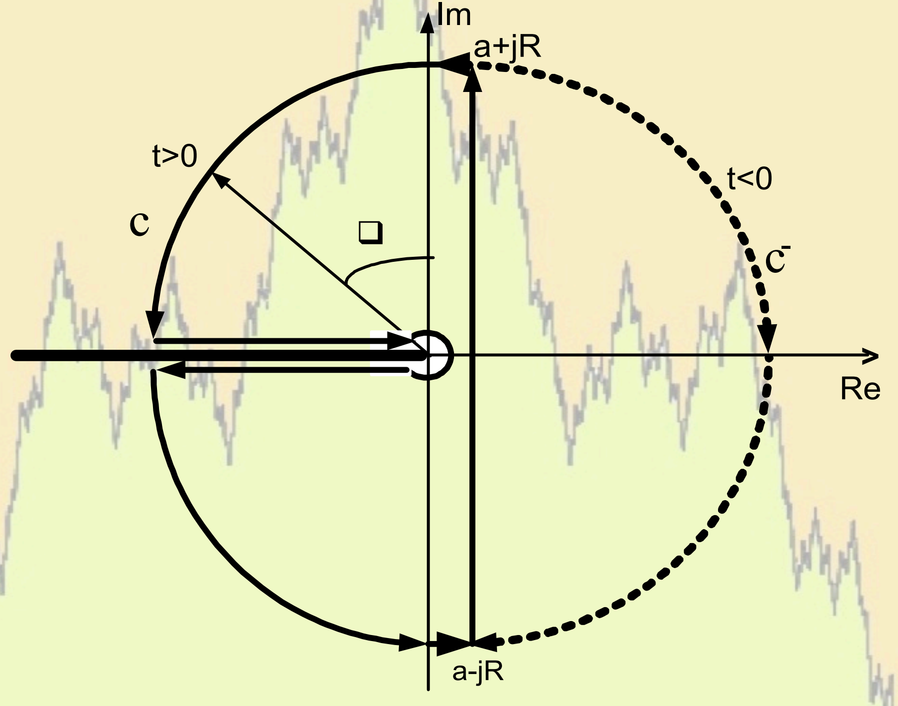
- How to realize the fractional order controller?
- How to tune the controller?

Controller with a fractor





Alternative partial fraction inversion



$$h(t) = \frac{p^{1/\alpha - 1}}{\alpha} e^{p^{1/\alpha} t} u(t) + \frac{1}{\pi} \int_0^{\infty} \frac{\sigma^{\alpha} \sin(\pi\alpha)}{\sigma^{2\alpha} - 2\sigma^{\alpha} p \cos(\pi\alpha) + p^2} e^{-\sigma t} \, d\sigma \; u(t)$$

Stability of a system

Consider the TF

$$G(s) = \frac{1}{s^{\alpha} - p}$$

The zero of the denominator, if it exists, is at $p^{1/\alpha}$.

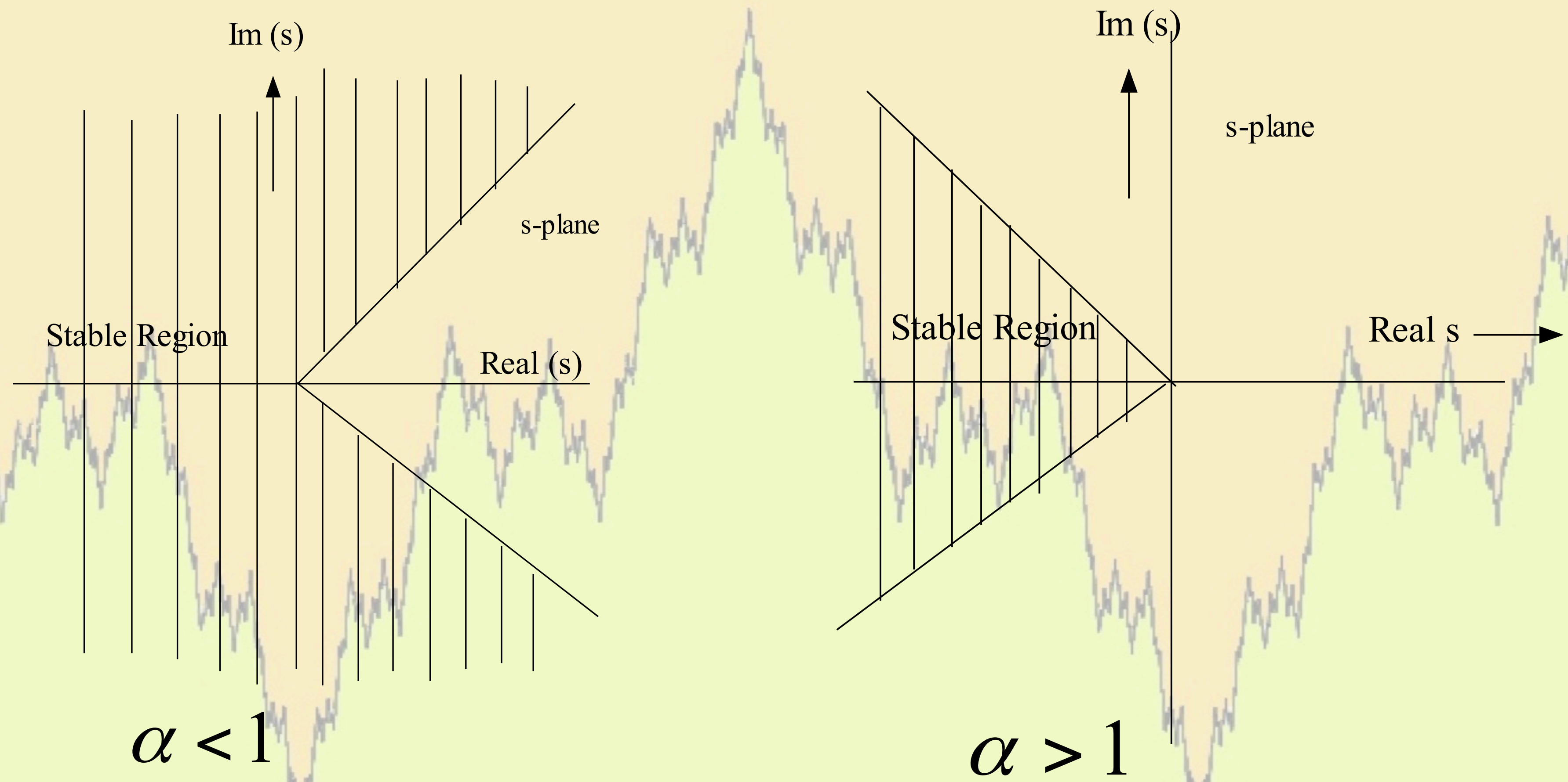
If $p=\rho e^{j\theta}$, then there is a pole if $|\theta/\alpha| \leq \pi$. If $|\theta/\alpha| > \pi$, <u>there is no pole.</u>

Conclusions:

$|\theta| > \pi\alpha$ **stability**

$$|\theta| \leq \pi\alpha \begin{cases} |\theta| < \pi\alpha/2 & \textbf{instability} \\ |\theta| > \pi\alpha/2 & \textbf{stability} \\ |\theta| = \pi\alpha/2 & \textbf{strict stability} \end{cases}$$

Stability region in s-plane



The stability region is not convex for α <1.

Initial conditions

- Pseudo-initial conditions:
  - » Riemann-Liouville
  - » Caputo
  - » Laplace transform
- The initial value problem;
- General approach;

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Pseudo-initial conditions

Riemann-Liouville

$$x_{RL}^{(\alpha)}(t) = D^{\alpha}[x(t)u(t)] - \sum_{i=0}^{n-1}[D^{\alpha-1-i}x(0)].\delta^{(i)}(t)$$

Caputo derivative

$$x_{C}^{(\alpha)}(t) = D^{\alpha}[x(t)u(t)] - \sum_{i=0}^{m-1}[D^{m-1-i}x(0)].\delta^{(i-m+\alpha)}(0)$$

Liouville derivative $$D_f^{\alpha}x(t)= \frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{t} x(\tau)u(\tau).(t-\tau)^{-\alpha-1}\,d\tau$$

$$LT[x_{RL}^{(\alpha)}(t)] = s^{\alpha}X(s) - \sum_{i=0}^{n-1}[D^{\alpha-1-i}x(0)].s^{i}$$

$$LT[x_{C}^{(\alpha)}(t)] = s^{\alpha}X(s) - \sum_{i=0}^{m-1}D^{m-i-1}x(0).s^{i-m+\alpha}$$

These "initial conditions" represent what lacks to the derivative to become a Liouville derivative

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The fractional jump formula

$$\varphi^{(\gamma_N)}(t) = [y(t).u(t)]^{(\gamma_N)} - \sum_{0}^{N-1} y^{(\gamma_i)}(0)\,\delta^{(\gamma_N-\gamma_i-1)}(t)$$

It is made continuous by subtracting $y^{(\gamma_N)}(0)\ u(t)$

The effects of the jumps are successively removed to obtain a continuous funtion

Jump formula – particular case

Making: $\gamma_n = n\gamma$

$$\varphi^{(n\gamma)}(t) = [y(t).u(t)]^{(n\gamma)} - \sum_{0}^{N-1} y^{(i\gamma)}(0)\delta^{((n-i)\gamma-1)}(t)$$

Main areas for research

1) Fractional control of engineering systems,
2) Fundamental explorations of the mechanical, electrical, and thermal constitutive relations and other properties of various engineering materials such as viscoelastic polymers, foam, gel, and animal tissues, and their engineering and scientific applications,
3) Advancement of Calculus of Variations and Optimal Control to fractional dynamic systems,
4) Fundamental understanding of wave and diffusion phenomenon, their measurements and verifications,
5) Analytical and numerical tools and techniques,
6) Bioengineering and biomedical applications,
7) Thermal modeling of engineering systems such as brakes and machine tools,
8) Image processing.

Where do we go to?

Fractional Discrete-Time Linear Systems

Fractional Systems on Time Scales

Fractional Vectorial Calculus and Classic Theorems: Gauss, Green, Stokes

Fractional Differential Geometry

…

Where do we go to?

•**EVERYWHERE**

Fractional Calculus:

the Calculus for the XXIth century

(Nishimoto)

Fractional Systems:

The XXIth Century Systems

(mdo)