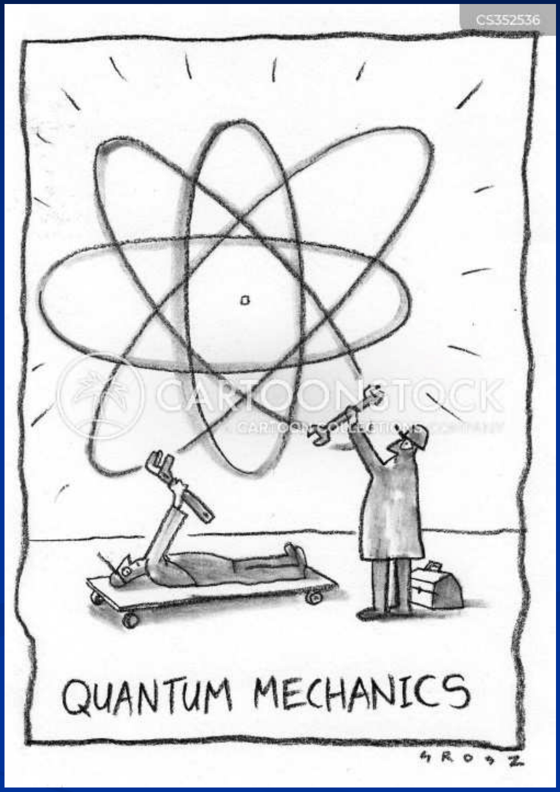
OnLine Estimation of Quantum Information Systems



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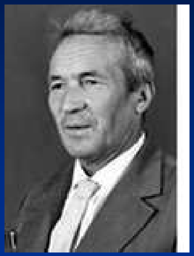
- Mark Balas is the Leland T. Jordan Professor of Dynamical Systems at the Texas A&M University. He has the following technical degrees: PhD in Mathematics, MS Electrical Engineering, MA Mathematics, and BS Electrical Engineering. He has held various positions in industry, academia, and government. Among his careers, he has been a university professor for over 45 years with University of Tennessee, RPI, MIT, University of Colorado-Boulder, University of Wyoming, Embry-Riddle Aeronautical University and has mentored 47 doctoral students to completion of their degrees. He has over 400 publications in archive journals, refereed conference proceedings and technical book chapters. He has been visiting faculty with the Institute for Quantum Information and the Control and Dynamics Division at the California Institute of Technology, the US Air Force Research Laboratory-Kirtland AFB, the NASA-Jet Propulsion Laboratory, the NASA Ames Research Center. He is a life fellow of the American Institute of Aeronautics and Astronautics (AIAA), a life fellow of the Institute of Electrical and Electronic Engineers (IEEE), and a fellow of the American Society of Mechanical Engineers (ASME). He is the recipient of the AIAA GNC Control Systems Heritage Lifetime Achievement award 2018. But, if he is ever well-known, it will be as the father of the prominent Denver Drum and Bass DJ known as Despise, who is his daughter Maggie; now Doctor Despise (Molecular Biology).

Quantum Probability vs Classical Probability



Andrei Kolmogorov

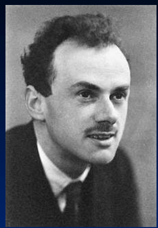
Event Space: X

Ω σ-algebra of subsets of X

Probability of event $A \equiv p(A): 0 \leq p(A) \leq 1$,

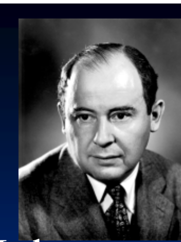
$$p(X) = 1, \& \, p(\Phi) = 0, \& \quad p(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} p(A_i) \text{ when } A_i \text{ disjoint}$$

<u>Bayes Theorem</u>: $p(A|B)p(B) = p(A \cap B) = p(B \cap A) = p(B|A)p(A)$



Paul Dirac

Quantum Basics: Quantum Probability



John
Von Neumann

__QuantumProbability:__

EventSpace : X complex (infinite-dimensional, separable) Hilbert Space

$X = \overline{span\{\phi_1, \phi_2, \phi_3, ...\}}$ orthonormal basis $(\phi_k, \phi_l) = \delta_{kl}$

Events $\equiv$ Closed Subspaces S of X (or their Projections)

$S_k \equiv span\{\phi_k\}$ basic subspace ("pure" states)

Mixed States: $x = \sum_{k=1}^{\infty} \underbrace{(x, \phi_k)\phi_k}_{P_k x} \quad \& \|x\|^2 = 1$

Superposition of Projections

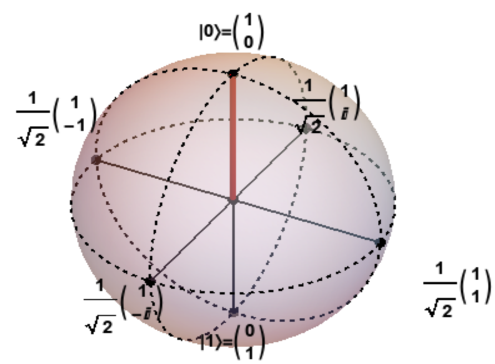
$$\sum_{k=1}^{\infty} P_k = I$$

__Quantum Probability:__

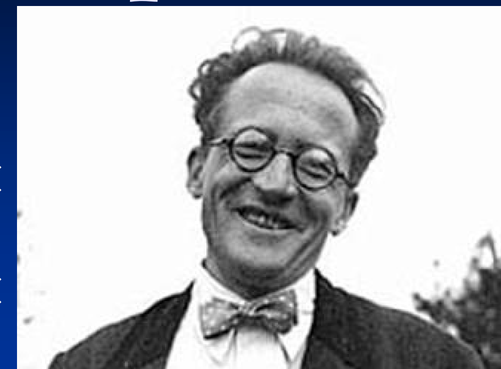
$p(x \in S_k) \equiv \|P_k x\|^2 = |(x, \phi_k)|^2 = |c_k|^2$

$Note: p(x \in S_k \mid x \in S_l) \equiv \|P_k P_l x\|^2 \neq \|P_k P_l x\|^2 = p(x \in S_l \mid x \in S_k)$

Unit Ball =Bloch Sphere



Dynamics: Schrodinger Wave Equation



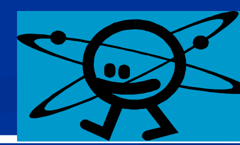
$\phi \in X$ complex Hilbert Space

$$i\hbar \frac{\partial \phi}{\partial t} = \underbrace{H_0}_{\text{Hamiltonian Energy Operator}} \phi \qquad \text{Discrete Spectrum } \sigma(H_0) = \{\lambda_k\}_{k=1}^{\infty}$$

$$\Rightarrow \phi(t) = \underbrace{U_0(t)}_{\text{Unitary Group}} \phi(0) = e^{-\frac{i}{\hbar}H_0 t}\phi(0) = \sum_{k=1}^{\infty} e^{-\frac{i\lambda_k}{\hbar}t}(\phi(0),\phi_k)\phi_k \text{ with } (\phi_k,\phi_l) = \delta_{kl}$$

$\therefore \|\phi(t)\|^2$ = Probability Distribution for the Energy

in the Quantum State ϕ(t) $\Rightarrow \|\phi(t)\| = \|\phi(0)\|$



$-\infty$

Marginally Stable

$\Rightarrow \therefore \|\phi(t)\|^2$ = Probability Distribution for the Energy

in the Quantum State ϕ(t)

$\Rightarrow \|\phi(t)\| = \|\phi(0)\|$

Dynamics: Schrodinger Wave Equation

$\phi \in X$ complex Hilbert Space

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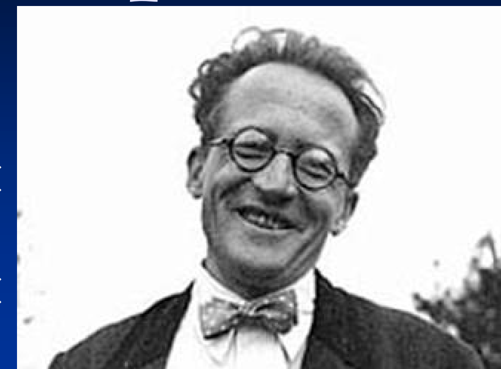
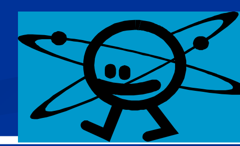




$-\infty$

Marginally Stable

$\Rightarrow \therefore \|\phi(t)\|^2$ = Probability Distribution for the Energy

in the Quantum State $\phi(t)$

$\Rightarrow \|\phi(t)\| = \|\phi(0)\|$

QuantumMeasurement

The Real Heisenberg

The interpretation of Quantum Measurement is still a controversial part of Quantum Theory

A quantum measurement is an entanglement with the environment (measuring device)

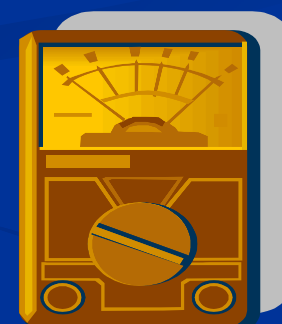
Entanglement

$$X = X_S \otimes X_M$$

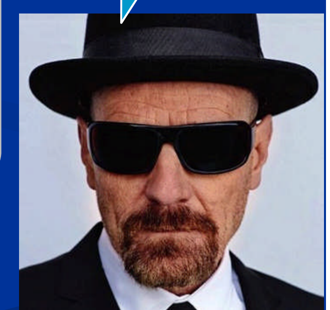
$$\phi = \sum_{k,l} \alpha_{kl} (\phi_k^S \otimes \phi_l^M) \neq h \otimes w$$

S

M



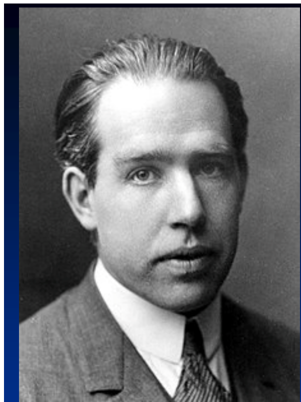
The Other Heisenberg



Back Action

Heisenberg Uncertainty Principle

$$(\Delta z)^2 (\Delta p)^2 \geq \left| (\underbrace{[z, p]}_{\frac{i\hbar}{2}} \phi, \phi) \right| = (\frac{\hbar}{2})^2 ; \hbar \approx 10^{-34}$$



Neils Bohr

Quantum Collapse:

Ontology vs Epistemology

Observable $A : X \xrightarrow[\text{self-adjoint}]{\text{bounded/unbounded}} X$

$$Ax = \sum_{k=1}^{\infty} \lambda_k \underbrace{(x, \phi_k)\phi_k}_{P_k x}$$

<u>Pure States</u>: ϕ_k eigenfunctions of A

Max Born

An <u>observation/measurement</u> of the observable A produces

a <u>collapse</u> of the wave function for a mixed state $\phi = \sum_{k=1}^{\infty} c_k \phi_k$

into one of the pure eigenstates ϕ_k $(A\phi_k = \lambda_k \phi_k)$ with probability $|c_k|^2$

Quantum Statistical Mechanics

Ensemble Behavior
from Multiple Experiments

Quantum Density Operators : $\rho \in \mathbb{C}^{NxN}$ with (Hilbert-Schmidt) inner product $(\rho_1, \rho_2) \equiv tr(\rho_1^* \rho_2)$

(These carry all the quantum probability information & are often thought of as quantum states)

Defining Properties: $\rho^* = \rho$(self-adjoint); $\rho \geq 0$(pos semi-definite); $tr\,\rho = 1$

Mixed State: $\rho = \sum_{k=1}^{N} p_k P_k$;

Pure State: $P_k \equiv (\phi_k, \bullet)\phi_k = \phi_k \phi_k^*$

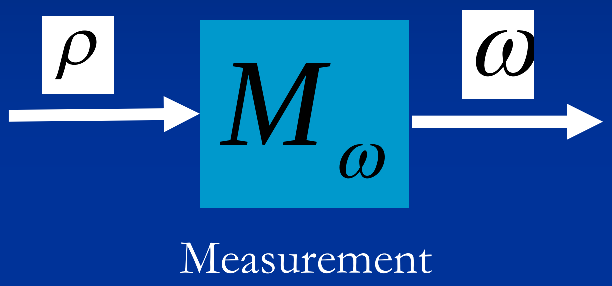
$$Dynamics: \frac{d\rho}{dt} = -i[H, \rho] = -i(\underbrace{H\rho - \rho H}_{L})$$,Quantum Master Equation

Ensemble Averages; Quantum Measurements:

y= $\langle C \rangle \equiv$ tr(Cρ)

Quantum Measurement

POVM=Positive Operator-Valued Measure



Measurement Operators

Outcomes

$$M \equiv \{M_\omega\}_{\omega\in\Omega}$$

$$M_\omega \text{ self-adjoint}; M_\omega \geq 0; \sum_{\omega\in\Omega} M_\omega = I \text{ (Complete)}$$

Probability of obtaining outcome ω

$$p_\rho^M(\omega) \equiv tr(\rho M_\omega) = tr(M_\omega^{\frac{1}{2}} \rho M_\omega^{\frac{1}{2}}) \geq 0$$

$$\Rightarrow \sum_{\omega\in\Omega} p_\rho^M(\omega) = \sum_{\omega\in\Omega} tr(\rho M_\omega) = tr\rho \underbrace{\sum_{\omega\in\Omega} M_\omega}_{I} = tr\rho = 1$$

Quantum Dynamical System

ρ_0 →

$$\text{Dynamics}: \begin{cases} \dfrac{d\rho}{dt} = -i[H,\rho] = -i(H\rho - \rho H) = -iL\rho\ , \\ \rho(0) = \rho_0 \end{cases}$$

Quantum Master Equation

Ensemble Averages; Quantum Measurements:

$$y = \langle C \rangle \equiv tr(C\rho)$$

→ y(t)

$$S \equiv \{\rho \in \mathbb{C}^{NxN} \mid \rho^* = \rho; \rho \geq 0; tr\rho = 1; tr\rho^2 \leq 1\}$$

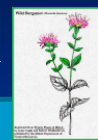
is a <u>bounded</u>, <u>closed</u>, <u>convex</u> subset of $\mathbb{C}^{NxN}$

Note: S is an <u>invariant set</u>: $\rho(0) \in \mathrm{S} \Rightarrow \rho(t) \in \mathrm{S}\ \forall t \geq 0$

"Once a quantum density, always a quantum density"

A Basic Online <u>Linear</u> Estimator

Quantum Density System



$$\begin{cases} \dfrac{\partial \rho}{\partial t} = -iL\rho; L \equiv [H, \rho] \\ y = trC\rho; \rho(0) = \rho_0 \end{cases}$$

y

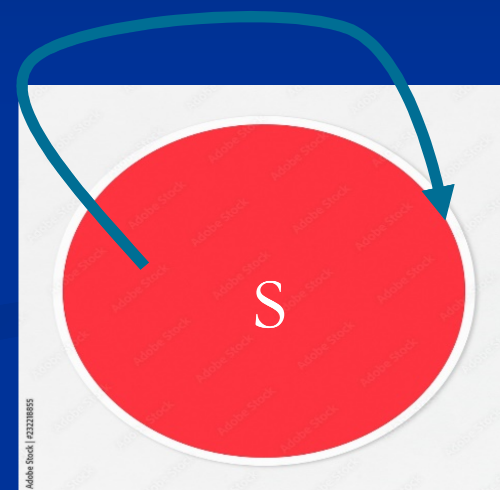
Linear Quantum Density Estimator

$$\begin{cases} \dfrac{\partial \hat{\rho}}{\partial t} = -iL\hat{\rho} + K(y - \hat{y}) \\ \hat{y} = C\hat{\rho}; \hat{\rho}(0) = \hat{\rho}_0 \end{cases}$$

$\hat{\rho}(t) = \rho(t) + e(t)$

$e(t) \xrightarrow[t \to \infty]{} 0$

with exponential rate



But $\hat{\rho}(t)$ <u>does not remain in</u> S

even tho it starts there $(\hat{\rho}(0) \in S)$

and it converges to $\rho(t) \in S$

The Set of All Quantum Density Operators

$S \equiv \{\rho \in \mathbb{C}^{NxN} \mid \rho^{*} = \rho; \rho \geq 0; tr\,\rho = 1; tr\,\rho^{2} \leq 1\} \subseteq$ Unit Ball in $\mathbb{C}^{NxN}$

<u>Theorem</u>: S is a <u>closed</u>, <u>convex</u> subset of $\mathbb{C}^{NxN}$, & S is <u>bounded</u> ($S \subseteq$ Unit Ball), where

S <u>closed</u> means: $\forall \{\rho_k\} \subseteq S \;\&\; \rho_k \xrightarrow[k \to \infty]{} \rho \Rightarrow \rho \in S$;

S <u>convex</u> means: $\forall \rho_1, \rho_2 \in S$, the straight line $\lambda \rho_1 + (1-\lambda)\rho_2 \in S$

Projection Operator for Closed Convex Sets in Hilbert Space

X Hilbert Space with S closed, convex $\subseteq X$.

$P_S : X \to S$:

$P_S x$ is the (metric) Projection of x onto S when

$\forall x \in X \quad \|x - P_S x\| = d(x, S) \equiv \min_{z \in S} \|x - z\|$

Properties of the Projection

1) $P_S(x)$ is defined $\forall x \in X$

2) $P_S(x) = x \Leftrightarrow x \in S$

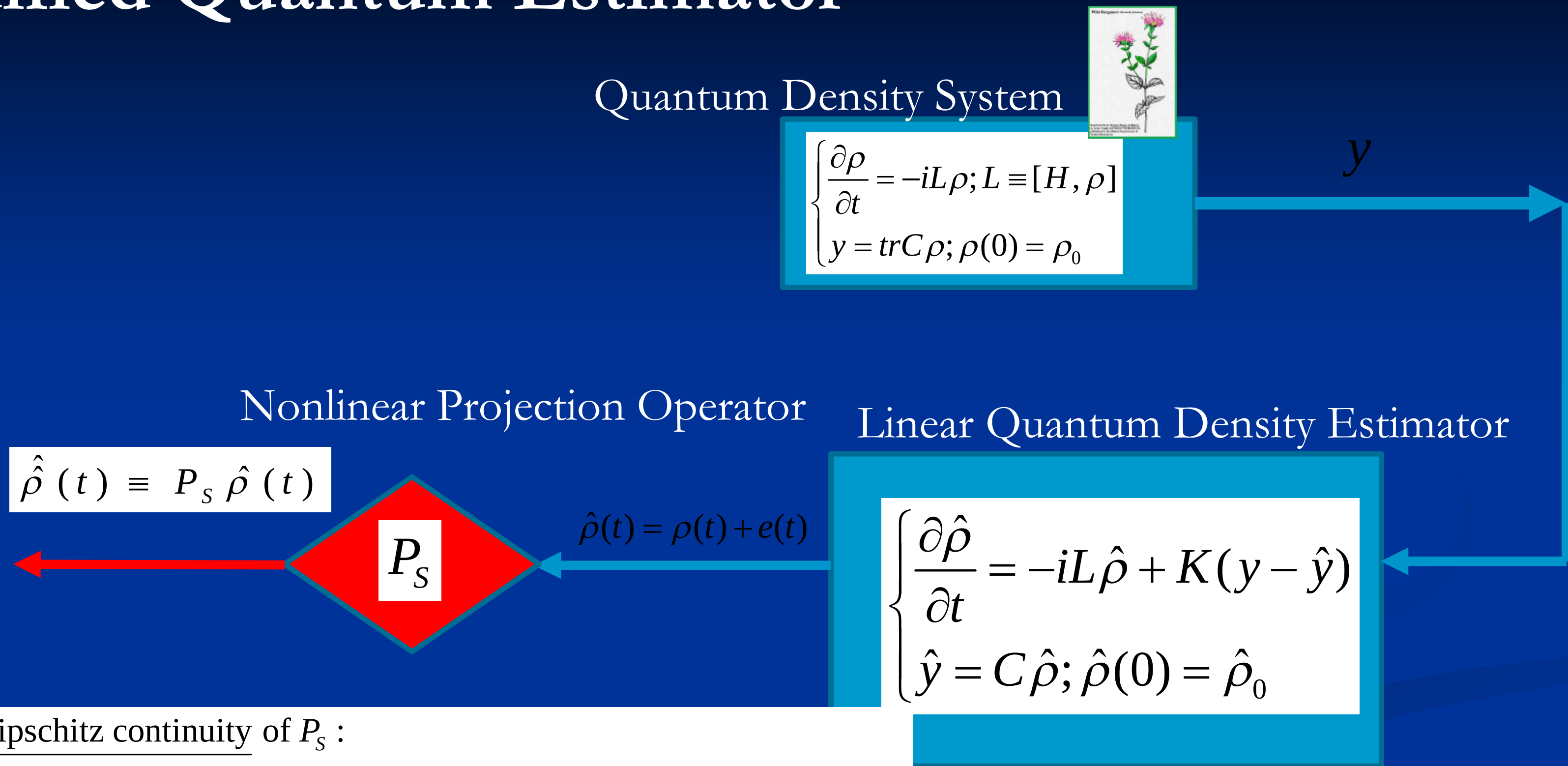
3) $P_S{}^2 = P_S\,(idempotent)$

4) $x_* = P_S x \Leftrightarrow \mathrm{Re}(\underbrace{x - x_*}_{Error}, z - x_*) \le 0 \;\; \forall z \in S$ ("Principle of Orthogonality, sorta")

5) P_S is Lipschitz Continuous, *i.e.* $\|P_S x - P_S y\| \le \|x - y\| \quad \forall x, y \in X$

But P_S is NOT Linear.

Modified Quantum Estimator



Using the Lipschitz continuity of P_S :

$$\left\| \hat{\hat{\rho}}(t) - \rho(t) \right\| = \left\| P_S \hat{\rho}(t) - \underbrace{P_S \rho(t)}_{\rho(t) \in S} \right\| \le \left\| \underbrace{\hat{\rho}(t) - \rho(t)}_{e(t)} \right\| = \left\| e(t) \right\|$$

where $e(t) \xrightarrow[t \to \infty]{} 0$ with exponential rate set by the original Linear Estimator

And $\hat{\hat{\rho}}(t) \equiv P_S \hat{\rho}(\mathrm{t})$ remains in S $\forall$t (and is a Quantum Density) even tho $\hat{\rho}(t)$ does not and it converges to $\rho(t) \in S$

Quantum Information Theory

<u>Classical: Shannon Entropy</u> $H(x) = \sum_{i=1}^{n} p_i \log p_i,$

"the average amount of information gained from learning the value of the random variable x"

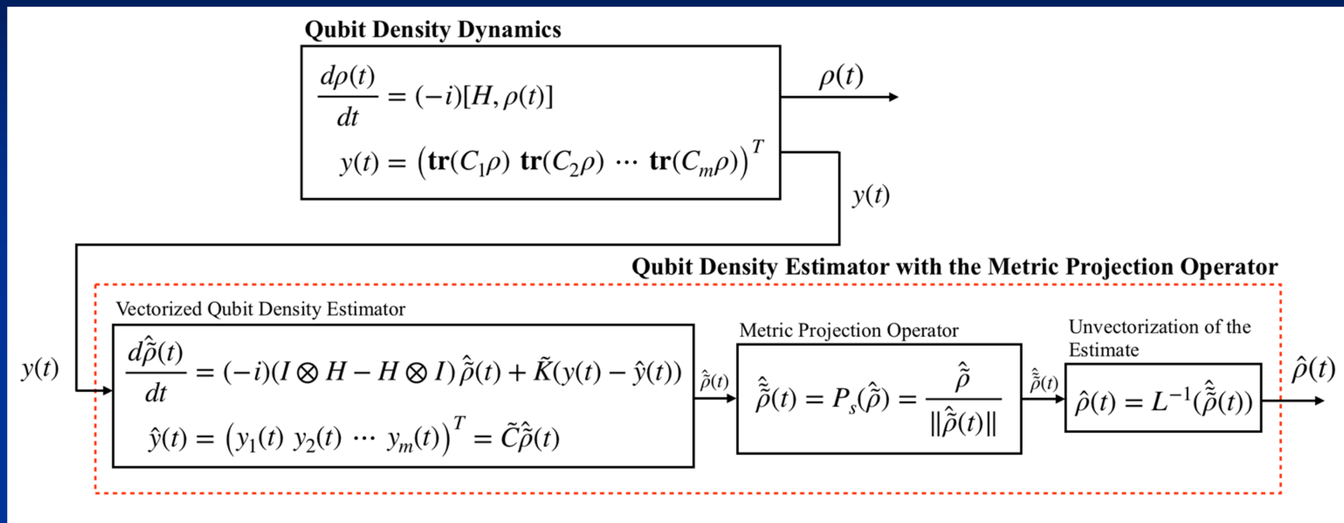
or "the average uncertainty before learning the value of x"

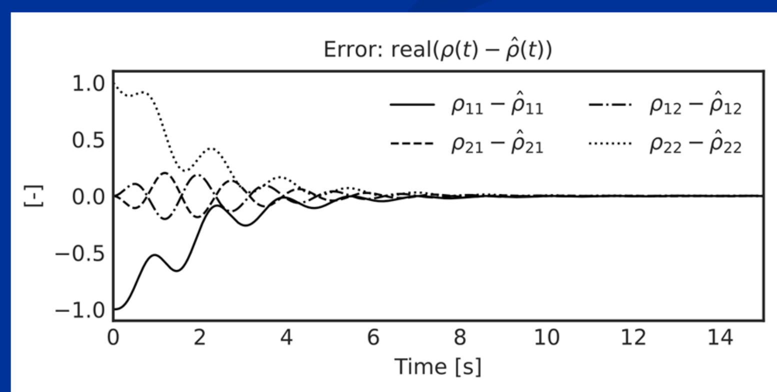
Quantum VonNeumann Entropy: $S(\rho) \equiv \text{-}tr(\rho \ \log \rho)$

$$\text{Theorem:} \left| S(\hat{\hat{\rho}}) - S(\rho) \right| \to 0 \ (\square \ \underline{te^{-\sigma t}}),$$

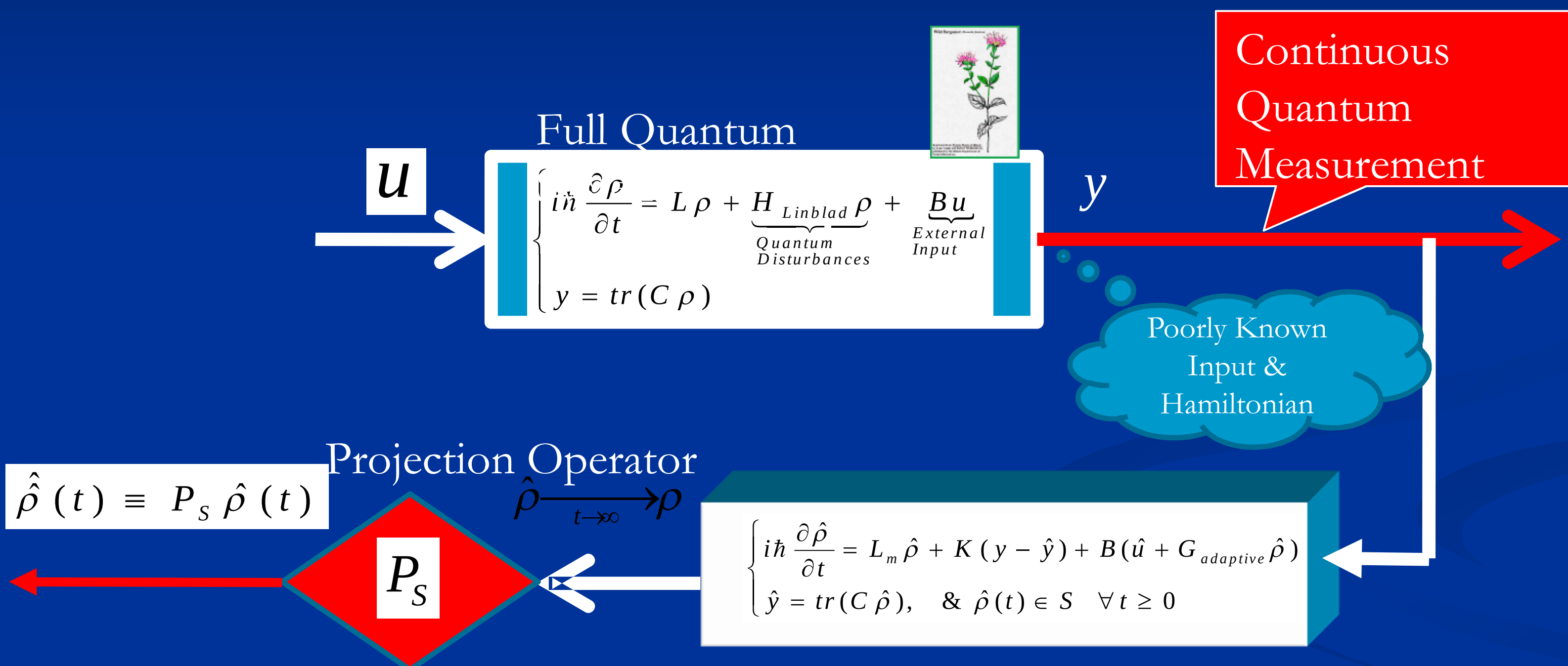
$$\text{when } \left\| \hat{\hat{\rho}} - \rho \right\|_{tr} \to 0 \ (\square \ e^{-\sigma t} \text{exponentially})$$

Qubit Estimator





Adaptive Quantum State Estimation in Hilbert Space



And $\hat{\hat{\rho}}(t) \equiv P_S\hat{\rho}(t)$ remains in S $\forall$t (and is a Quantum Density) even tho $\hat{\rho}(t)$ does not and it converges to $\rho(t) \in S$

Quantum Cognition

Whoa?!

QuantumProbability:

EventSpace : X complex

(infinite-dimensional, separable) Hilbert Space

$X = \overline{span\{\phi_1, \phi_2, \phi_3, ...\}}$ orthonormal basis $(\phi_k, \phi_l) = \delta_{kl}$

Events $\equiv$ Closed Subspaces S of X (or their Projections)

$S_k \equiv span\{\phi_k\}$ basic subspace

Mixed States: $x = \sum_{k=1}^{\infty} \underbrace{(x, \phi_k)\phi_k}_{P_k x} \quad \& \|x\|^2 = 1$

Quantum Probability:

$p(x \in S_k) \equiv \|P_k x\|^2 = |(x, \phi_k)|^2 = |c_k|^2$

$Note: p(x \in S_k \mid x \in S_l) \equiv \|P_k P_l x\|^2 \neq \|P_k P_l x\|^2 = p(x \in S_l \mid x \in S_k)$

Model of Human Decision-Making

NSF Proposal: A Quantum Approach to Human Cognition and the Autonomy Conundrum in Self Driving Vehicles, James Hubbard and Mark Balas

References

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Tristan Griffith, James Hubbard, and Mark Balas, *A Modal Approach to the Space-Time Dynamics of Cognitive Biomarkers*, Springer-Verlag, 2021 ISBN: 978-3-031-23528-3

"We don't know where we are stupid until we stick our necks out"
……………………..Richard Feynman