Zero-Sum Games with Distributionally Robust Chance Constraints

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The Seventeenth International Conference on Internet and Web Applications and Services

ICIW, Porto, Portugal

June $26^{th} - 30^{th}$, 2022

Outline of the talk

1. Introduction
2. Game model
3. Main results
4. Conclusion

Introduction

- We consider a two-player zero-sum game with random linear chance constraints whose distributions are known to belong to moments based uncertainty sets.
- We show that a Saddle Point Equilibrium problem is equivalent to a primal-dual pair of Second-Order Cone Programs.
- The game with chance constraints can be used in various applications, e.g., risk constraints in portfolio optimization, resource constraints in stochastic shortest path problem, renewable energy aggregators in the local market.

- $X = \{x \in \mathbb{R}^m \mid C^1 x = d^1, x \geq 0\}$ is the strategy set of player 1.
- $Y = \{y \in \mathbb{R}^n \mid C^2 y = d^2, y \geq 0\}$ is the strategy set of player 2.
- $u : X \times Y \to \mathbb{R}$ is a payoff function associated to the game where:

$$u(x, y) = x^T G y + g^T x + h^T y, \tag{1}$$

- Player 1 is interested in maximizing u and player 2 is interested in minimizing u.
- The strategies of player 1 satisfy the following random linear constraints:

$$(a_k^1)^T x \leq b_k^1, \;\; k = 1, 2, \ldots, p, \tag{2}$$

whilst the strategies of player 2 satisfy the following random linear constraints

$$(a_l^2)^T y \geq b_l^2, \;\; l = 1, 2, \ldots, q. \tag{3}$$

- In some situations, we lost full knowledge of the true distribution of the random variables defined in (2) and (3). For this reason, we assume that these distributions belong to some uncertainty sets.
- Using the worst case approach, the random linear constraints (2) and (3) can be formulated as distributionally robust chance constraints given by:

$$\inf_{F_k^1 \in \mathcal{D}_k^1} \mathbb{P}\left((a_k^1)^T x \leq b_k^1\right) \geq \alpha_k^1, \ \forall \ k = 1, ..., p, \tag{4}$$

$$\inf_{F_l^2 \in \mathcal{D}_l^2} \mathbb{P}\left((-a_l^2)^T y \leq -b_l^2\right) \geq \alpha_l^2, \ \forall \ l = 1, ..., q, \tag{5}$$

Game model

Two players zero-sum game

- The feasible strategy sets of player 1 and player 2 are given by:

$$S^1_{\alpha^1} = \Big\{x \in X| \inf_{F^1_k \in \mathcal{D}^1_k} \mathbb{P}\{(a^1_k)^T x \leq b^1_k\} \geq \alpha^1_k, \ \forall \ k = 1, ..., p\Big\}, \quad (6)$$

$$S^2_{\alpha^2} = \Big\{y \in Y| \inf_{F^2_l \in \mathcal{D}^2_l} \mathbb{P}\{(-a^2_l)^T y \leq -b^2_l\} \geq \alpha^2_l, \ \forall \ l = 1, ..., q\Big\}. \quad (7)$$

- We denote the game with the strategy set $S^1_{\alpha^1}$ for player 1 and the strategy set $S^2_{\alpha^2}$ for player 2 by Z_α.

Definition 1

A strategy pair $(x^*, y^*) \in S^1_{\alpha^1} \times S^2_{\alpha^2}$ is called an SPE of the game Z_α at $\alpha = (\alpha^1, \alpha^2) \in [0,1]^p \times [0,1]^q$, if

$$u(x, y^*) \leq u(x^*, y^*) \leq u(x^*, y), \ \forall \ x \in S^1_{\alpha^1}, \ y \in S^2_{\alpha^2}.$$

Main results

Choices of uncertainty sets

- **Polytopic uncertainty set**
- **Ellipsoidal uncertainty set**

- **Polytopic uncertainty set**

$$\mathcal{D}_k^i\left(\mu_k^i, \Sigma_k^i\right) = \left\{ F_k^i \;\middle|\; \begin{array}{c} E_{F_k^i}\left[a_k^i\right] \in U_{\mu_k^i} \\ COV_{F_k^i}[a_k^i] \in U_{\Sigma_k^i} \end{array} \right\}, \qquad (8)$$

where $U_{\mu_k^i} = Conv(\mu_{k1}^i, \mu_{k2}^i, \ldots, \mu_{kM}^i)$ and $U_{\Sigma_k^i} = Conv(\Sigma_{k1}^i, \Sigma_{k2}^i, \ldots, \Sigma_{kM}^i)$.

- **Ellipsoidal uncertainty set**

$$\mathcal{D}_k^i(\mu_k^i, \Sigma_k^i) = \left\{ F_k^i \;\middle|\; \begin{array}{l} \left(\mathbb{E}_{F_k^i}[a_k^i] - \mu_k^i\right)^\top \left(\Sigma_k^i\right)^{-1} \\ \times \left(\mathbb{E}_{F_k^i}[a_k^i] - \mu_k^i\right) \le \gamma_{k1}^i, \\ \\ COV_{F_k^i}[a_k^i] \preceq \gamma_{k2}^i \Sigma_k^i \end{array} \right\}, \qquad (9)$$

Existence of an SPE

Theorem 2

Consider the game Z_α, where the distributions of the random constraint vectors of 2 players belong to either the polytopic uncertainty set or ellipsoidal uncertainty set. Then, there exists an SPE of the game for all $\alpha \in (0,1)^p \times (0,1)^q$.

Characterization of an SPE

- (x^*, y^*) is an SPE for the game Z_α if and only if

$$x^* \in \arg\max_{x \in S^1_{\alpha^1}} \min_{y \in S^2_{\alpha^2}} u(x,y), \tag{19}$$

$$y^* \in \arg\min_{y \in S^2_{\alpha^2}} \max_{x \in S^1_{\alpha^1}} u(x,y). \tag{20}$$

Theorem 3

(x^*, y^*) is an SPE for the game Z_α if and only if there exist auxiliary variables $\nu^1, \delta^1, \lambda^1$ and $\nu^2, \delta^2, \lambda^2$ such that:

- *$y, \nu^1, \delta^1, \lambda^1$ are the optimal solutions of the SOCP D_y.*
- *$x, \nu^2, \delta^2, \lambda^2$ are the optimal solutions of the SOCP D_x.*

Main results

$$\min_{y,\ \nu^1,\ \delta^1_{k,j},\ \lambda^1_{k,j}} \quad h^T y + (\nu^1)^T d^1 + \sum_{k=1,\dots,p} \sum_{j=1}^{M} \lambda^1_{k,j} b^1_k$$

s.t.

$$(i)\ Gy - \sum_{k=1,\dots,p} \sum_{j=1}^{M} \left(\lambda^1_{k,j}\mu^1_{k,j} + \left(\Sigma^1_{k,j}\right)^{\frac{1}{2}} \delta^1_{k,j}\right)$$

$$-\,(C^1)^T \nu^1 + g \le 0,$$

$$(ii)\ \ -(\mu^2_{lj})^T y + \kappa_{\alpha^2_l} ||(\Sigma^2_{lj})^{\frac{1}{2}} y|| \le -b^2_l,$$

$$\forall\ j = 1, 2, \dots, M,\ \ l = 1, ..., q,$$

$$(iii)\ ||\delta^1_{k,j}|| \le \kappa_{\alpha^1_k} \lambda^1_{k,j},\ \ \lambda^1_{k,j} \ge 0,$$

$$\forall\ k = 1, ..., p,\ \ j = 1, 2 \dots, M,$$

$$(iv)\ C^2 y = d^2,\ \ y_s \ge 0,\ \forall\ s = 1, 2, \dots, n. \qquad (D_y)$$

Main results

$$\max_{x,\ \nu^2,\ \delta^2_{l,j},\ \lambda^2_{l,j}} \ g^T x + (\nu^2)^T d^2 + \sum_{l=1,\dots,q} \sum_{j=1}^{M} \lambda^2_{l,j} b^2_l$$

$$\text{s.t. } (i)\ G^T x - \sum_{l=1,\dots,q} \sum_{j=1}^{M} \left(\lambda^2_{l,j}\mu^2_{l,j} + \left(\Sigma^2_{l,j}\right)^{\frac{1}{2}} \delta^2_{l,j}\right)$$

$$-(C^2)^T \nu^2 + h \geq 0,$$

$$(ii)\ (\mu^1_{kj})^T x + \kappa_{\alpha^1_k} ||(\Sigma^1_{kj})^{\frac{1}{2}} x|| \leq b^1_k,$$

$$\forall\ j = 1, 2, \dots, M,\ \ k = 1, ..., p,$$

$$(iii)\ ||\delta^2_{l,j}|| \leq \kappa_{\alpha^2_l} \lambda^2_{l,j},\ \ \lambda^2_{l,j} \geq 0,\ \forall\ \ l = 1, ..., q,\ \ j = 1, 2, \dots M$$

$$(iv)\ C^1 x = d^1,\ \ x_r \geq 0,\ \forall\ r = 1, 2, \dots, m. \qquad (D_x)$$

Conclusion

- We show the existence of a mixed strategy SPE for a two-player distributionally robust zero-sum chance-constrained game under three different uncertainty sets based on first two moments.
- We derive tractable reformulations of the distributionally robust chance constraints of each player in two case, polytopic uncertainty and ellipsoidal uncertainty.
- The tractable reformulation of the zero-sum game problem with different payoff structure, as well as the uncertainty sets other than the ones considered in the paper could be an interesting area for the future research.

Thank you for your attention.