Global Stability of Positive Different Fractional Orders Nonlinear Feedback Systems

by Tadeusz Kaczorek and Łukasz Sajewski

Łukasz SAJEWSKI

Faculty of Electrical Engineering

Bialystok University of Technology

Bialystok, Poland

l.sajewski@pb.edu.pl

POLITECHNIKA BIAŁOSTOCKA

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Resume of the presenter

ORCID: 0000-0003-1098-5571
RG profile
Research team
Ongoing project

Lukasz Sajewski (born 1981, Poland) received his MSc degree in electrical engineering from the Bialystok University of Technology in 2006 and then started his PhD studies. In 2007 he began lecturing in the field of microprocessor techniques and programmable logic devices. In 2009 he received a PhD degree in electrical engineering from the Bialystok University of Technology and the DSc degree in February 2018. At present, he works at the Faculty of Electrical Engineering in a position of the Professor and the Head of Department of Automatic Control and Robotics. His main scientific interests involve control theory, especially descriptor, positive, continuous-discrete and fractional systems. He has published over 50 scientific papers and one book. His research interests also cover the aspects of using programmable logic devices and programmable logic controllers in automatic control of industrial processes. In these fields he is a supervisor of MSc theses.

Outline

- Preliminaries
- Fractional different orders nonlinear feedback systems with positive linear parts
- Numerical example
- Concluding remarks

Preliminaries

Consider the fractional continuous-time linear system

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = Ax(t) + Bu(t), \qquad (1a)$$

$$y(t) = Cx(t), \qquad (1b)$$

where $x(t) \in \Re^{n}$, $u(t) \in \Re^{m}$, $y(t) \in \Re^{p}$ are the state, input and output vectors, $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$.

The following Caputo definition of the fractional derivative of α order will be used

$$_{0}D_{t}^{\alpha} f(t) = \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{\dot{f}(\tau)}{(t-\tau)^{\alpha}} d\tau, \quad 0 < \alpha < 1, \qquad (2)$$

where $\dot{f}(\tau) = \frac{df(\tau)}{d\tau}$ and $\Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} dt$, $\mathrm{Re}(x) > 0$ is the Euler gamma function.

Definition 1. The fractional system (1) is called (internally) positive if $x(t) \in \Re_+^n$ and $y(t) \in \Re_+^p$, $t \geq 0$ for any initial conditions $x(0) \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 1. The fractional system (1) is positive if and only if

$$A \in M_n,\ B \in \Re_+^{n \times m},\ C \in \Re_+^{p \times n}. \tag{3}$$

Definition 2. The fractional positive linear system (1) is called asymptotically stable (and the matrix A Hurwitz) if

$$\lim_{t \to \infty} x(t) = 0 \text{ for all } x(0) \in \Re_+^n. \tag{4}$$

The positive fractional system (1) is asymptotically stable if and only if the real parts of all eigenvalues s_k of the matrix A are negative, i.e. $\mathrm{Re}\, s_k < 0$ for $k = 1, \ldots, n$.

Theorem 2. The positive fractional system (7) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{5}$$

are positive, i.e. $a_i > 0$ for $i = 0,1,\ldots,n-1$.

2) There exists strictly positive vector $\lambda = [\lambda_1 \quad \cdots \quad \lambda_n]$, $\lambda_k > 0$, $k = 1,\ldots,n$ such that

$$A\lambda < 0 \text{ or } \lambda^T A < 0. \tag{6}$$

The transfer matrix of the system (1) is given by

$$T(s^\alpha) = C[I_n s^\alpha - A]^{-1} B. \tag{7}$$

Now, consider the fractional linear system with two different fractional order

$$\begin{bmatrix} \dfrac{d^{\alpha} x_1(t)}{dt^{\alpha}} \\ \dfrac{d^{\beta} x_2(t)}{dt^{\beta}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), \tag{8a}$$

$$y(t) = [C_1 \quad C_2] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{8b}$$

where $0 < \alpha, \beta < 1$, $x_1(t) \in \Re^{n_1}$ and $x_2(t) \in \Re^{n_2}$ are the state vectors, $A_{ij} \in \Re^{n_i \times n_j}$, $B_i \in \Re^{n_i \times m}$, $C_i \in \Re^{p \times n_i}$; $i, j = 1,2$; $u(t) \in \Re^m$ is the input vector and $y(t) \in \Re^p$ is the output vector.

Initial conditions for (8) have the form

$$x_1(0) = x_{10},\ x_2(0) = x_{20} \text{ and } x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}. \tag{9}$$

Definition 3. The fractional system (8) is called positive if $x_1(t) \in \Re_+^{n_1}$ and $x_2(t) \in \Re_+^{n_2}$, $t \geq 0$ for any initial conditions $x_{10} \in \Re_+^{n_1}$, $x_{20} \in \Re_+^{n_2}$ and all input vectors $u \in \Re_+^m$, $t \geq 0$.

Theorem 3. The fractional system (8) for $0 < \alpha < 1$; $0 < \beta < 1$ is positive if and only if

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in M_N, \ \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \Re_+^{N \times m},$$

$$C = [C_1 \quad C_2] \in \Re_+^{p \times n} (N = n_1 + n_2). \tag{10}$$

Theorem 4. The positive fractional system (8) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficients of the characteristic polynomial

$$\det[I_n s - \overline{A}] = s^n + \overline{a}_{n-1} s^{n-1} + \ldots + \overline{a}_1 s + \overline{a}_0 \tag{11}$$

are positive, i.e. $\overline{a}_i > 0$ for $i = 0,1,\ldots,n-1$.

2) There exists strictly positive vector $\lambda = [\lambda_1 \quad \cdots \quad \lambda_n]$, $\lambda_k > 0$, $k = 1,\ldots,n$ such that

$$\overline{A}\lambda < 0 \text{ or } \lambda^T \overline{A} < 0. \tag{12}$$

Theorem 5. The solution of the equation (8a) for $0<\alpha<1;\ 0<\beta<1$ with initial conditions (9) has the form

$$x(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}=\Phi_0(t)x_0+\int_0^t M(t-\tau)u(\tau)d\tau, \tag{13}$$

where

$$\begin{aligned} M(t)&=\Phi_1(t)B_{10}+\Phi_2(t)B_{01}=\begin{bmatrix} \Phi_{11}^1(t) & \Phi_{12}^1(t) \\ \Phi_{21}^1(t) & \Phi_{22}^1(t) \end{bmatrix}\begin{bmatrix} B_1 \\ 0 \end{bmatrix}+\begin{bmatrix} \Phi_{11}^2(t) & \Phi_{12}^2(t) \\ \Phi_{21}^2(t) & \Phi_{22}^2(t) \end{bmatrix}\begin{bmatrix} 0 \\ B_2 \end{bmatrix} \\ &=\begin{bmatrix} \Phi_{11}^1(t)B_1+\Phi_{12}^2(t)B_2 \\ \Phi_{21}^1(t)B_1+\Phi_{22}^2(t)B_2 \end{bmatrix}=\begin{bmatrix} \Phi_{11}^1(t) & \Phi_{12}^2(t) \\ \Phi_{21}^1(t) & \Phi_{22}^2(t) \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{aligned} \tag{14a}$$

and

$$\Phi_0(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha+l\beta+1)},\ \Phi_1(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{(k+1)\alpha+l\beta-1}}{\Gamma[(k+1)\alpha+l\beta]},\ \Phi_2(t)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}T_{kl}\frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma[k\alpha+(l+1)\beta]} \tag{14b}$$

$$T_{kl}=\begin{cases} I_n & \text{for } k=l=0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & \text{for } k=1, l=0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & \text{for } k=0, l=1 \\ T_{10}T_{k-1,l}+T_{01}T_{k,l-1} & \text{for } k+l>1 \end{cases} \tag{14c}$$

Note that, if $\alpha = \beta$, then from (13) we have

$$\Phi_0\big|_{\alpha=\beta}(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{15}$$

The transfer matrix of the system (8) is given by

$$T(s^\alpha, s^\beta) = \overline{C}\left[\begin{bmatrix} I_{n_1} s^\alpha & 0 \\ 0 & I_{n_2} s^\beta \end{bmatrix} - \overline{A}\right]^{-1} \overline{B}. \tag{16}$$

Fractional nonlinear feedback systems with positive linear parts

Consider the nonlinear feedback system shown in Figure 1, which consists of the positive linear part, the nonlinear element with characteristic $u = f(e)$ and the positive scalar feedback.

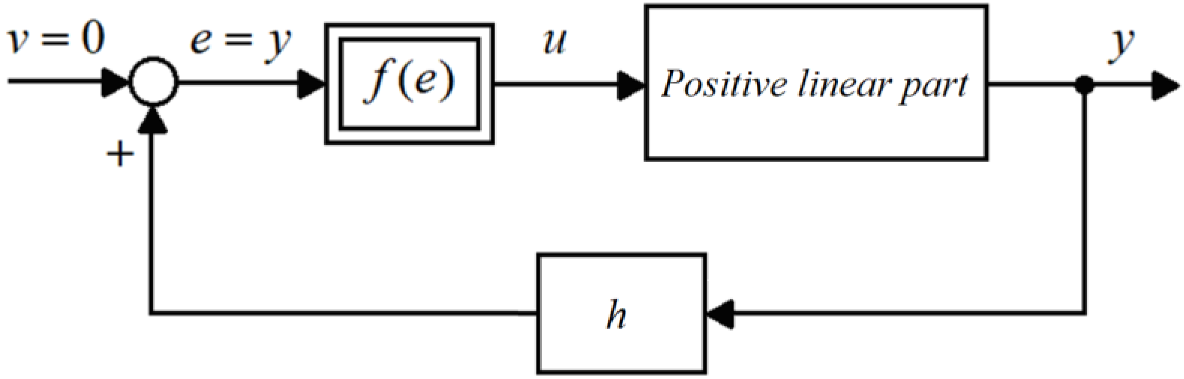

Figure 1. The nonlinear feedback system.

The positive linear part is described by the equations

$$\begin{bmatrix} \dfrac{d^{\alpha} x_1(t)}{dt^{\alpha}} \\ \dfrac{d^{\beta} x_2(t)}{dt^{\beta}} \end{bmatrix} = \overline{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \overline{B} u(t),$$
$$y(t) = \overline{C} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{17}$$

where $0 < \alpha, \beta < 1$, $x_1 = x_1(t) \in \Re^{n_1}$ and $x_2 = x_2(t) \in \Re_{+}^{n_2}$ are the state vectors, $u = u(t) \in \Re$ is the input vector, $y = y(t) \in \Re$ is the input vector, matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ for $p = m = 1$ are defined by (10).

The characteristic of the nonlinear element is shown in Figure 2 and it satisfies the condition

$$0 < f(e) < ke, \quad 0 < k < \infty. \tag{18}$$

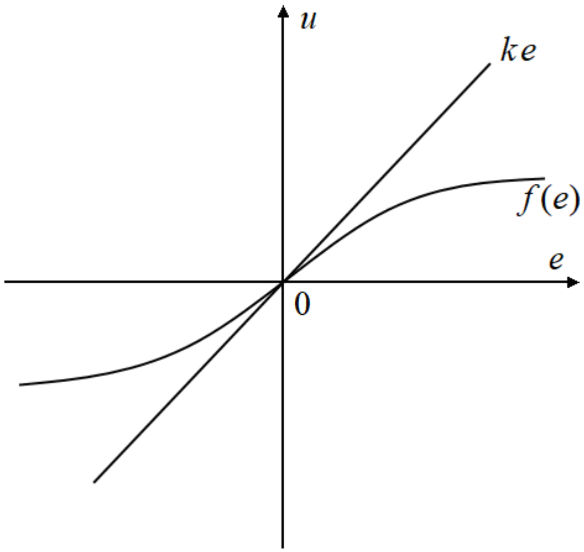

Figure 2. Characteristic of the nonlinear element.

It is assumed that, the positive linear part is asymptotically stable (the matrix $\bar{A} \in M_n$ is Hurwitz).

Definition 4. The nonlinear positive system is called globally stable if it is asymptotically stable for all nonnegative initial conditions $\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \in \Re_+^n$.

The following theorem gives sufficient conditions for the global stability of the positive nonlinear system.

Theorem 6. The nonlinear system consisting of the positive linear part, the nonlinear element satisfying the condition (18) and the positive scalar feedback h is globally stable if the matrix

$$\overline{A} + kh\overline{B}\overline{C} \in M_n \tag{19}$$

is asymptotically stable (Hurwitz matrix).
Matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ are given by (10).

Numerical example

Example 1. Consider the nonlinear system with the positive linear part with the matrices

$$A = \begin{bmatrix} -3 & 0.5 & 0.2 & 0.1 \\ 1 & -2 & 0.2 & 0.3 \\ 0.2 & 0.3 & -5 & 0.4 \\ 0.3 & 0.4 & 0.5 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0.2 \\ 0.6 \\ 0.4 \end{bmatrix}, \quad C = [0.2 \quad 0.4 \quad 0.5 \quad 0.3],$$

$$h = 0.5, \quad \alpha = 0.4, \quad \beta = 0.6, \quad n_1 = n_2 = 2, \tag{20}$$

the nonlinear element satisfying the condition (18) and the positive feedback with gain h.

Find k satisfying (19) for which the nonlinear system is globally stable for $h = 0.5$.

Using (14) and (17) for $h = 0.5$ we obtain

$$\hat{A} = \overline{A} + kh\overline{B}\,\overline{C} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + kh\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}[C_1 \quad C_2]$$

$$= \begin{bmatrix} -3+0.05k & 0.5+0.1k & 0.2+0.125k & 0.1+0.075k \\ 1+0.02k & -2+0.04k & 0.2+0.05k & 0.3+0.03k \\ 0.2+0.06k & 0.3+0.12k & -5+0.15k & 0.4+0.09k \\ 0.3+0.04k & 0.4+0.08k & 0.5+0.1k & -4+0.06k \end{bmatrix}. \qquad (21)$$

The characteristic polynomial of the matrix (21) has the form

$$\det(I_4 s - \hat{A}) = s^4 + (14 - 0.3k)s^3 + (70.05 - 3.31k)s^2 + (146.39 - 11.99k)s + (104.64 - 14.28k) \qquad (22)$$

and its coefficients are positive which implies that the nonlinear system with (20) is globally stable for $k < 7.33$.

Remark 1. The determinant of the matrix (21) has the form

$$\det(\hat{A}) = 104.64 - 14.28k \tag{23}$$

and it is equal to zero for $k = 7.33$.

Concluding remarks

- The global stability of continuous-time different fractional orders nonlinear feedback systems witch positive linear parts and positive scalar feedbback has been investigated
- New sufficient conditions for the global stability of this class of positive nonlinear systems are established
- The effectiveness of these new stability conditions has been demonstrated on simple example of positive nonlinear different orders system
- The considerations can be extended to discrete-time standard fractional different orders nonlinear systems with positive linear parts and scalar feedbacks
- An open problem is an extension of the considerations to nonlinear different orders fractional systems with interval matrices of their positive linear parts

Thank you for your attention