Modeling transient perturbations
Waves in heterogeneous and time-varying media

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Damian works at the intersection of photonics, applied physics, and applied mathematics. This interdisciplinary work led him to develop a new numerical scheme to solve hyperbolic PDEs with space-time-varying coefficients, and propose novel photonic integrated circuits (PICs) with applications in many branches of science and engineering.

He has a B. Sc. in Physics from the National Autonomous University of Mexico (UNAM), and M.Sc. (2010) and Ph.D. (2014) in Electrical Engineering from King Abdullah University of Science and Technology (KAUST)

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What happens when materials vary in time?

Space-time's geometry and Lagrange equations

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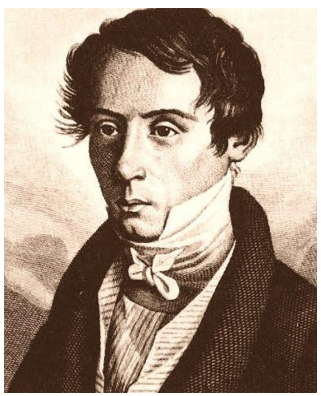
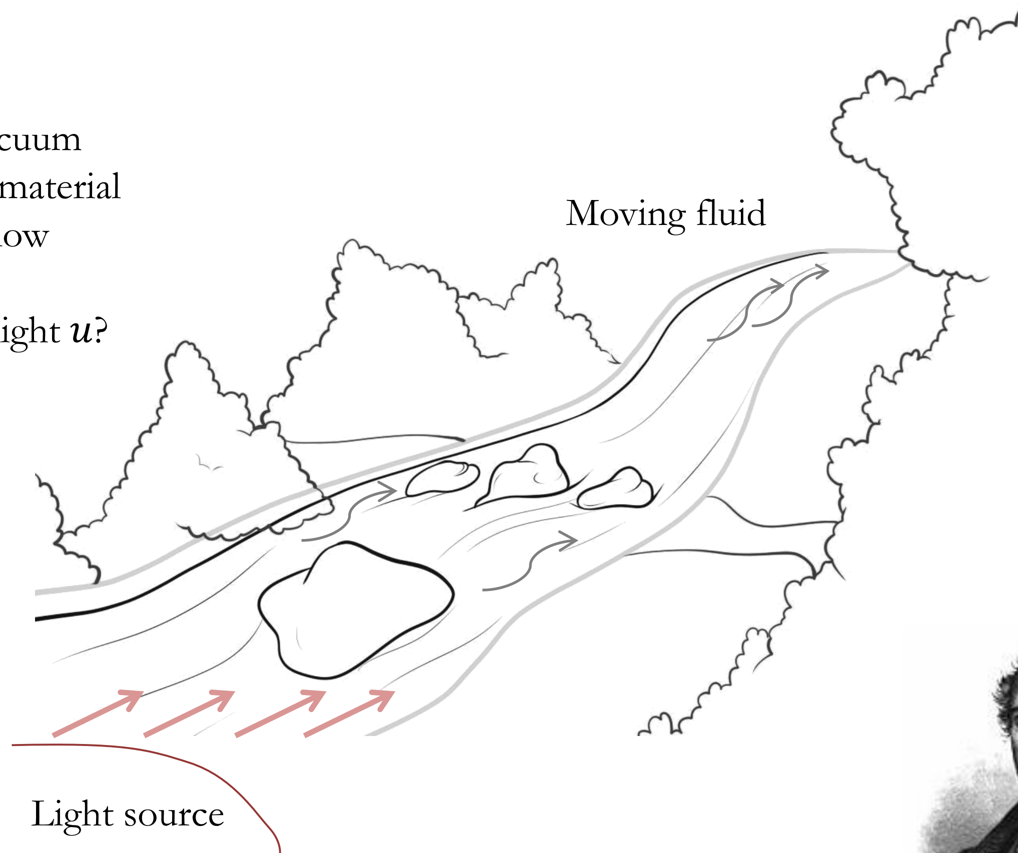
Fresnel drag factor

c: speed of light in vacuum
n: refractive index of material
v: speed of material flow

What is the speed of light u?



Augustin Fresnel.
Source: Wikimedia Commons



Wave propagation speed in a moving material

$$u \approx \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right) = u_o + w$$

Francois Arago.
Source: Wikimedia Commons

Waves are dragged by the moving media!

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The speed of light in a medium is given by the index of refraction, n as

$$u = \frac{c}{n}$$

If the medium moves at speed v then the velocity transforms to

$$u' = \frac{v + \frac{c}{n}}{1 + \frac{v}{cn}}$$

If $v \ll c$ the difference between u' and u is

$$\Rightarrow u' - u = \frac{v + \frac{c}{n}}{1 + \frac{v}{cn}} - \frac{c}{n} = \frac{v(1 - n^{-2})}{1 + \frac{v}{cn}} \approx v\left(1 - \frac{1}{n^2}\right)$$

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How to find the path of massless particles?

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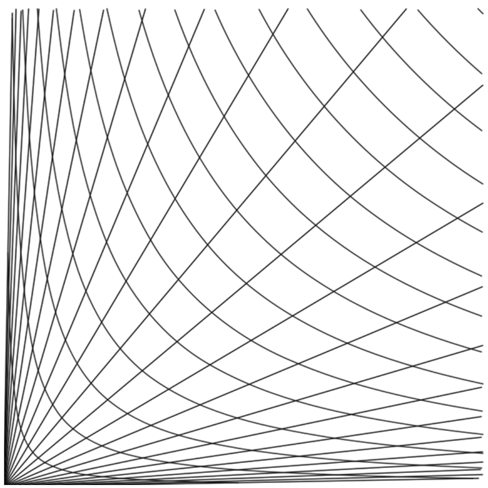
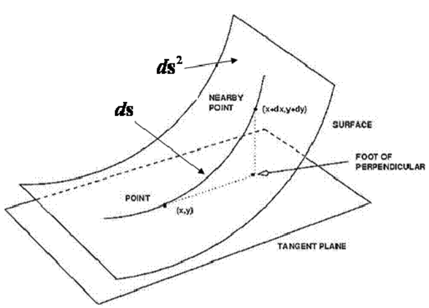
Lagrange equations





Let every point $\vec{r}$ in space be $\{x^0, x^1, x^2, x^3\}$, and the metric $ds^2 = g_{ii}dx^i dx^i,\ g = diag(c, -1, -1, -1)$.

Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\left[g_{ij}\frac{\partial x^i}{\partial \tau}\frac{\partial x^j}{\partial \tau}\right].$$

Solve Lagrange-Euler equations for the photon path,

$$\ddot{x}^i = \frac{(c\dot{x}^0)^2}{n^3}\frac{\partial n}{\partial x^i}$$

$$\ddot{x}^0 = \frac{(c\dot{x}^0)}{n}\left(c\dot{x}^0\frac{\partial n}{\partial x^0} + 2\dot{x}^i\frac{\partial n}{\partial x^i}\right).$$

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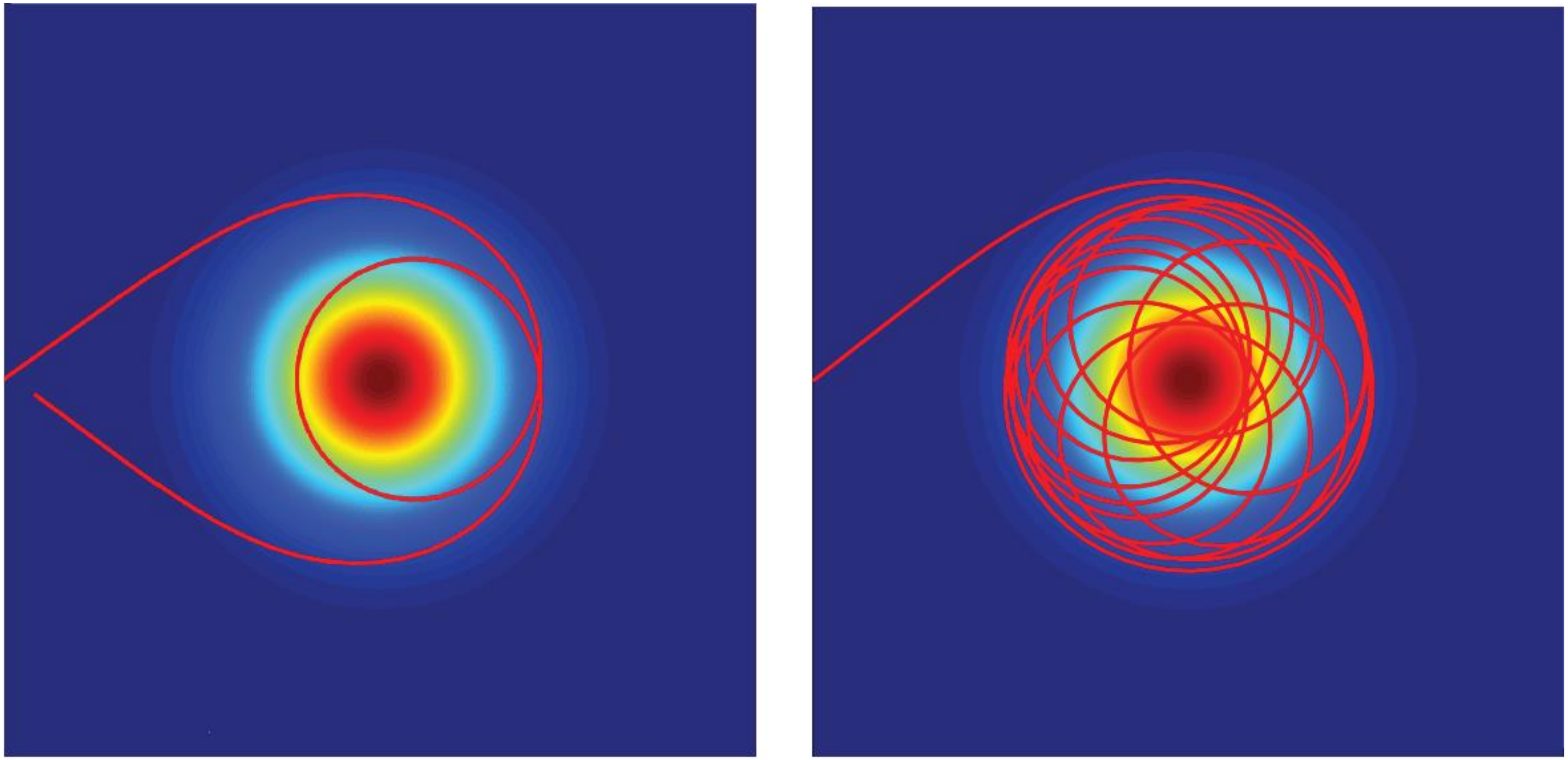
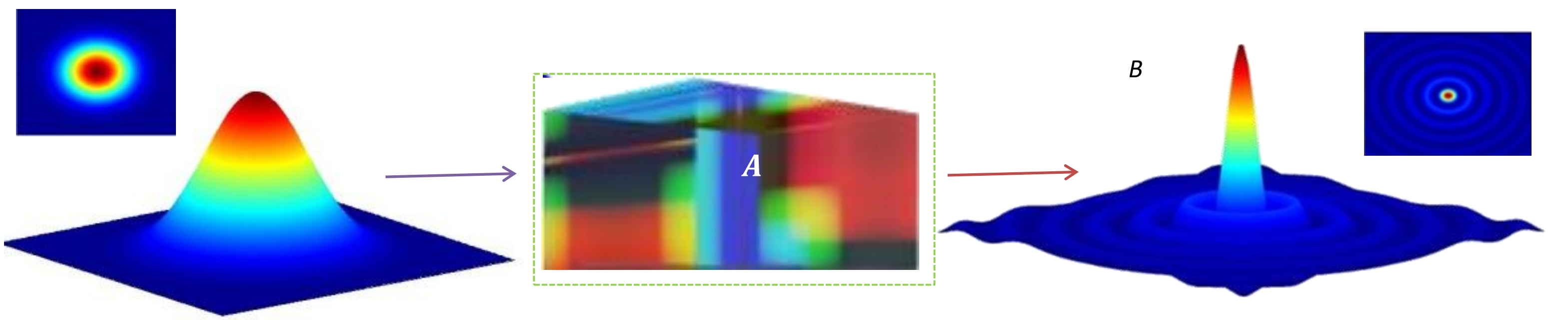
Mimicking nature



Mimicking celestial mechanics. Refraction in heterogeneous media with radial symmetry [San-Rom´an-Alerigi et al., 2013].



Gauss beam profile

J_0-Gauss-Bessel beam profile

Wave phenomena

Maxwell's equations in a charge- and current-free space,

$$\vec{D}_t - \nabla \times \vec{H} = 0, \tag{1a}$$

$$\vec{B}_t + \nabla \times \vec{E} = 0. \tag{1b}$$

With constitutive relations

$$\vec{D} = \vec{\zeta}_e\left(\bar{\varepsilon}, \vec{E}\right) \equiv \vec{\zeta}_e\left(\bar{\eta}_e, \vec{q}_e\right), \tag{2a}$$

$$\vec{B} = \vec{\zeta}_h\left(\bar{\mu}, \vec{H}\right) \equiv \vec{\zeta}_h\left(\bar{\eta}_h, \vec{q}_h\right), \tag{2b}$$

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Let

$$\vec{q} = \begin{pmatrix} E \\ H \end{pmatrix}, \qquad \vec{f} = \begin{pmatrix} H \\ E \end{pmatrix}$$

And define

$$\bar{\kappa} = \begin{pmatrix} \partial_E \zeta_e & 0 \\ 0 & \partial_H \zeta_h \end{pmatrix}, \qquad \vec{\psi} = \begin{pmatrix} -\partial_\varepsilon \zeta_e \, \partial_t \varepsilon \\ -\partial_\mu \zeta_h \, \partial_t \mu \end{pmatrix}.$$

where $\bar{\kappa}$ is the capacity function, $\vec{\psi}$ is the source term, and $\vec{f}$ is the flux.

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General hyperbolic PDE

We can rewrite Maxwell's equations as:

$$\bar{\kappa}(\vec{q}, x, t) \cdot \vec{q}(x, t)_t + \vec{f}(\vec{q})_x = \vec{\psi}(\vec{q}, x, t)$$

Incorporates heterogeneous, time-varying and anisotropic materials…

Many wave phenomena can be described by this equation!

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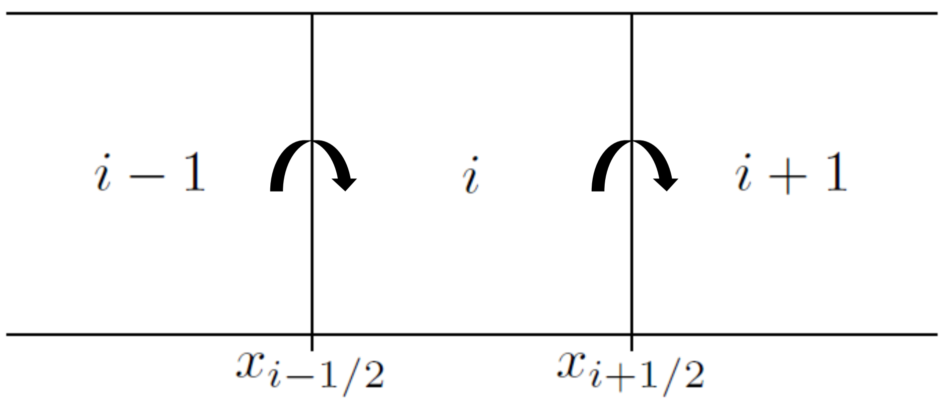
Solution

Flux differences between adjacent cells results in a Riemann problem, where the fluctuations at $x_{i-\frac{1}{2}}$ can be approximated in term of f-waves, the jump difference at the given interface.



$$\frac{\partial Q_i}{\partial t} = -\frac{1}{\bar{K}_i \Delta x}\left(\mathcal{A}^+ \Delta q_{i-\frac{1}{2}} + \mathcal{A}^- \Delta q_{i+\frac{1}{2}} + \mathcal{A}\Delta q_i - \Delta x \Psi_i \left(q^R_{i-\frac{1}{2}}, q^L_{i+\frac{1}{2}}, t\right)\right).$$

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To integrate use the ten-stage fourth-order strong-stability-preserving Runge-Kutta scheme described in [Ketcheson, 2008]. For reconstruction use fifth-order WENO reconstruction [Shu, 2009].

Algorithm, at every Runge-Kutta stage

1. Set cell averages of the capacity $\bar{K}_i^n$
2. Reconstruction, using fifth-order WENO compute the piecewise elements of q to get states $q^R_{i-\frac{1}{2}}, q^L_{i+\frac{1}{2}}$
3. Solve the Riemann problem with initial states $(q^L_{i-\frac{1}{2}}, q^R_{i+\frac{1}{2}})$ to compute the fluctuations, $\mathcal{A}^{\pm}\Delta q_{i-\frac{1}{2}}$
4. Calculate the total fluctuation $\mathcal{A}\Delta q_i$, use states $q^L_{i+\frac{1}{2}}, q^R_{i-\frac{1}{2}}$
5. Set the cell averages of the source Ψ_i^n and subtract
6. Compute $\partial Q_i/\partial t$ using the semidiscrete scheme

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Oscillating Media

Consider a flowing medium such that

$$\varepsilon_r = \mu_r = \eta = \eta_o + \delta\eta \sin\omega\, t$$

And the initial right-moving pulse as initial condition

$$E = H = q = q_p \exp - \left(\frac{x - x_o}{\sigma^2}\right)^2$$

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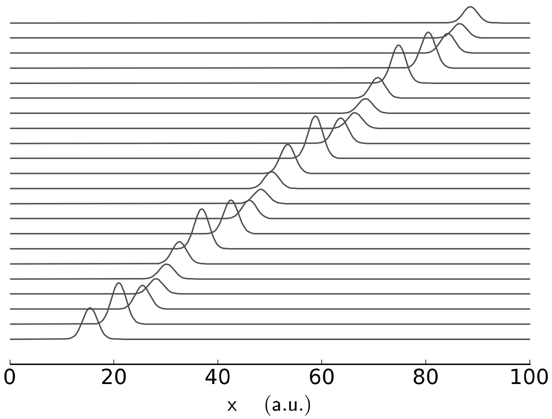
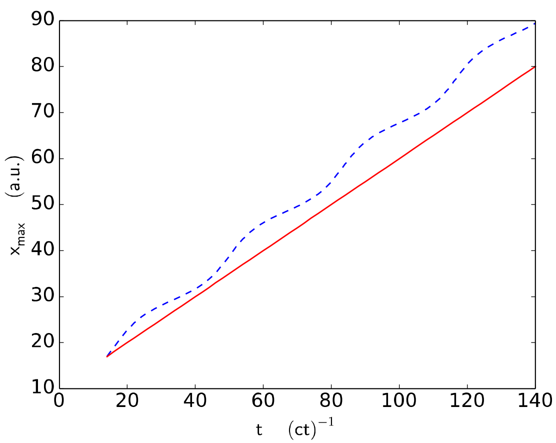
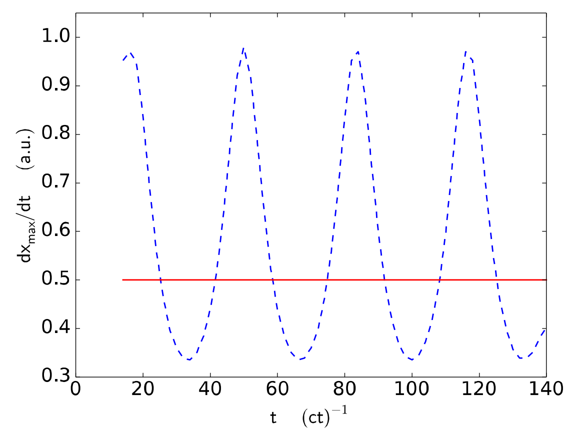


(a) Field magnitude



(b) Position of peak



(c) Velocity of peak

(a) Staggered plot of field intensity (black) and refractive index (green); (b) trace of peak; and (c) velocity of peak

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Flowing medium

Consider a flowing medium such that

$$\varepsilon_r = \mu_r = \eta = \eta_o + \delta\eta \exp - \left(\frac{x - x_o - v\,t}{\sigma^2} \right)^2$$

And the initial right-moving pulse as initial condition

$$E = H = q = q_p \exp - \left(\frac{x - x_o}{\sigma^2} \right)^2$$

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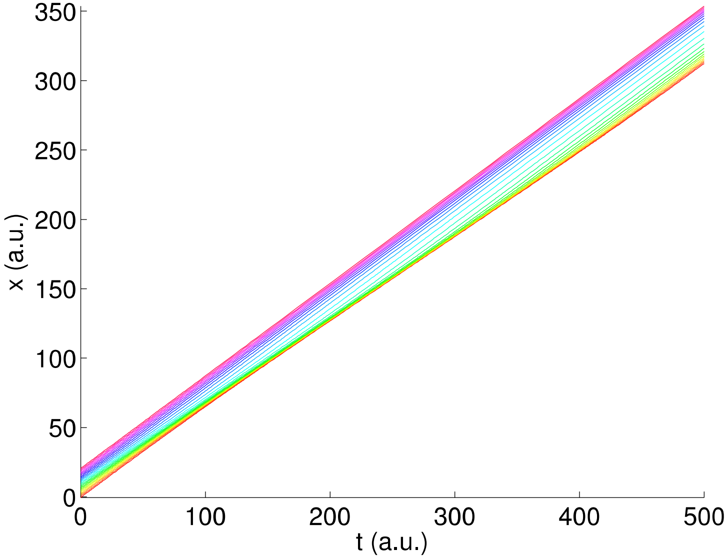
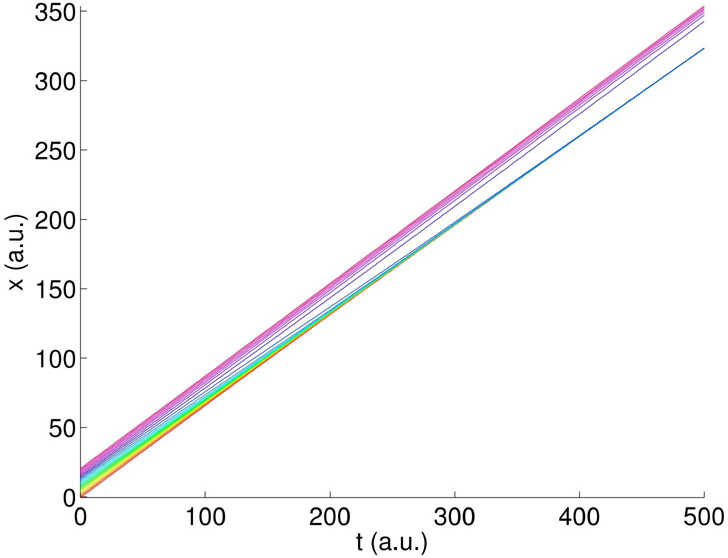


(a) $v = 0.59$



(b) $v = 0.63$

Characteristic of the solution as seen by an external observer

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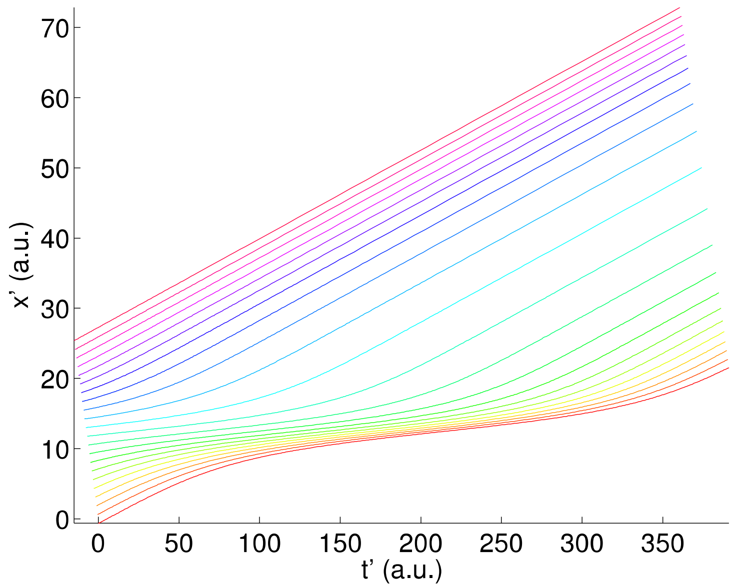
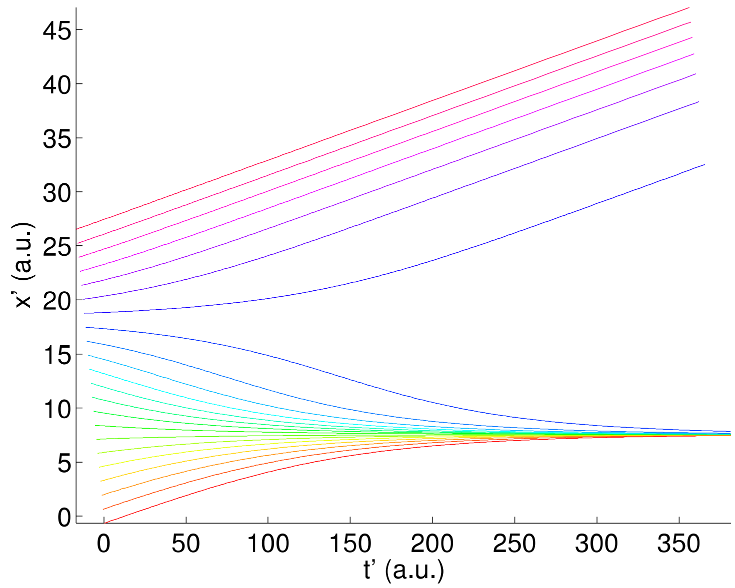


(a) $v = 0.59$



(b) $v = 0.63$

Characteristic of the solution as seen by a co-moving observer

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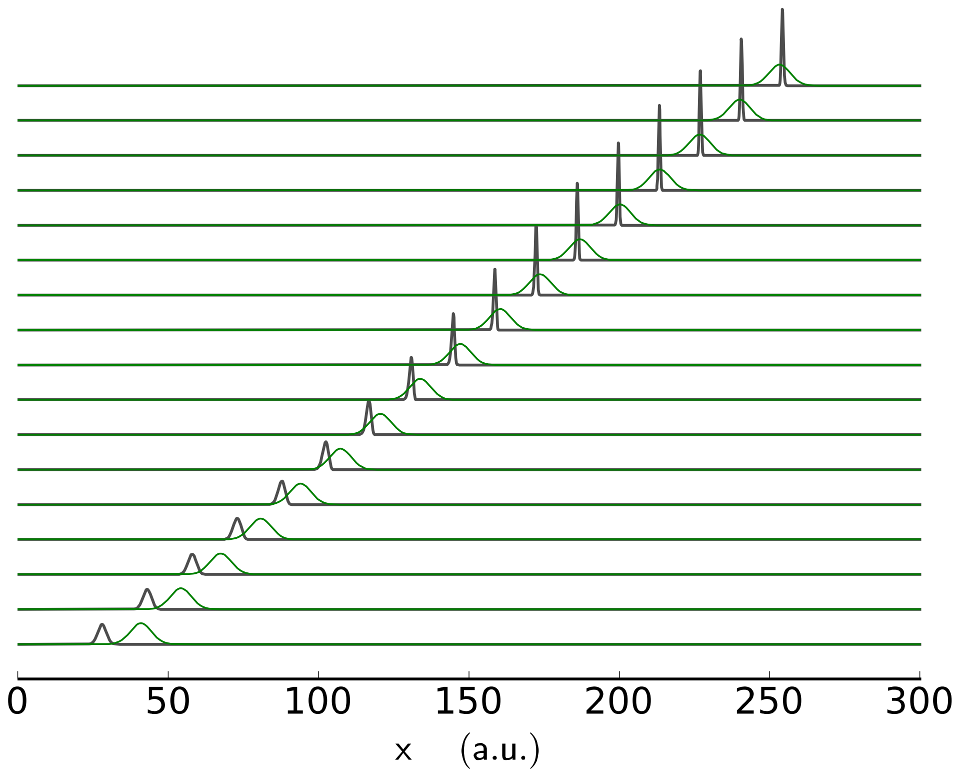
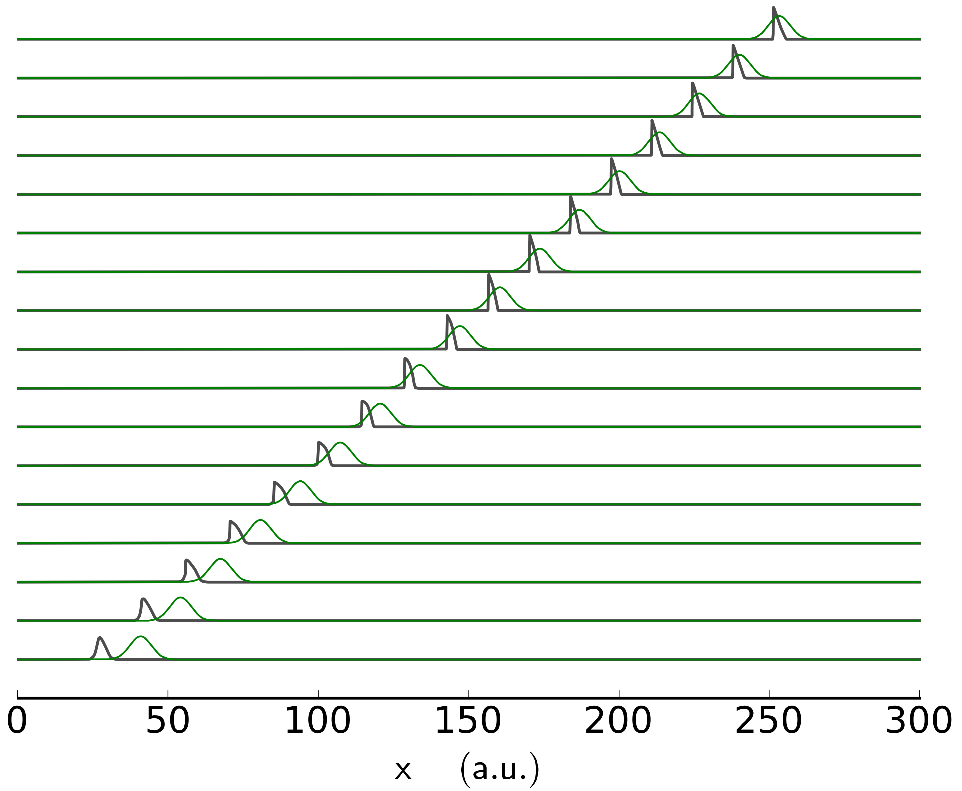


Staggered plot of field intensity (black) and refractive index (green); the latter being a Gaussian-like perturbation to the refractive index moving with velocity $v = 0.59$

Adding nonlinearity



Staggered plot of field intensity (black) and refractive index (green); the latter being a Gaussian-like perturbation to the refractive index moving with velocity $v = 0.59$ and nonlinear $\chi^{(3)} = 0.1$

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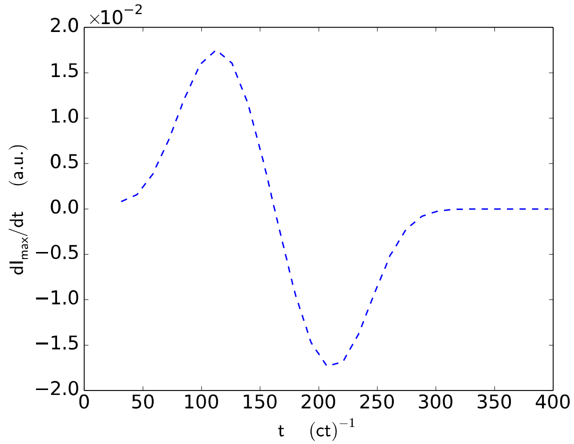
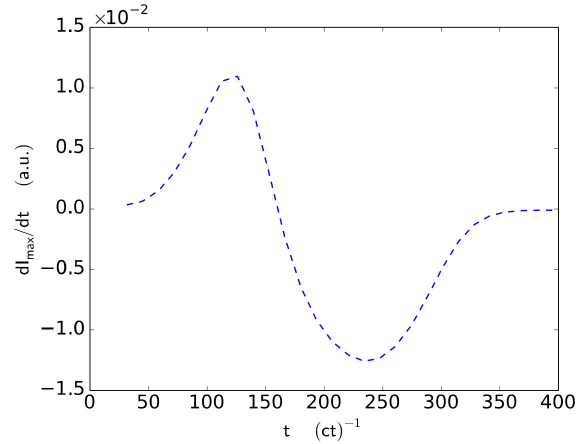
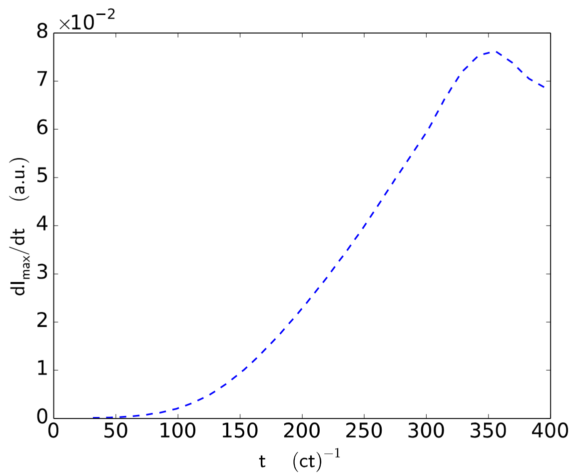


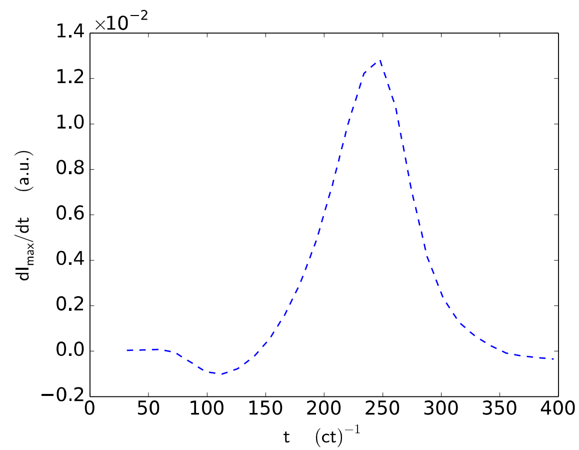
(a) $v = 0.55,\ \chi^{(3)} = 0$



(b) $v = 0.55,\ \chi^{(3)} = 0.1$



(c) $v = 0.61,\ \chi^{(3)} = 0$



(d) $v = 0.61,\ \chi^{(3)} = 0.1$

Rate of change with respect to time for the maximum intensity as seen from the laboratory frame; here the medium is a Gaussian-like perturbation to moving with velocity $v = 0.61$ and nonlinear $\chi^{(3)} = 0.1$

Consider a flowing medium such that

$$\kappa_i = \eta_o + \delta\eta \exp - \left(\frac{x - x_o - v\,t}{\sigma_x^2}\right)^2 - \left(\frac{y - y_o}{\sigma_y^2}\right)^2$$

And the initial right-moving pulse as initial condition

$$\begin{aligned} q_0(x,0) &= 0.0, \\ q_1(x,0) &= g(y,y_o)\; q_o(x,x_o,\sigma), \\ q_2(x,0) &= g(y,y_o)\; q_o(x,x_o,\sigma), \\ g(y) &= \cos\frac{(y-y_o)\,\pi}{L_y}, \end{aligned}$$

where L_y is the simulation length in the y direction.

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Vibrating media

Consider a vibrating medium with

$$\varepsilon_r = \mu_r = n = \eta_o + \eta_w \sin(\omega t) \cos(\theta_x x) \sin(\theta_y y)$$

And an initial Gaussian pulse located at the center

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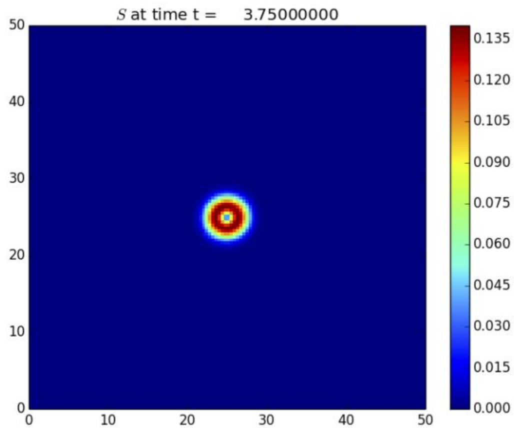
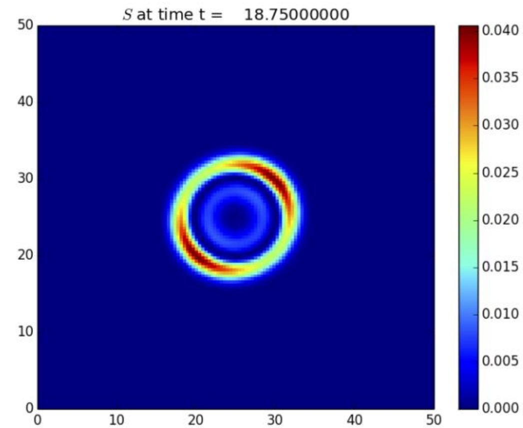
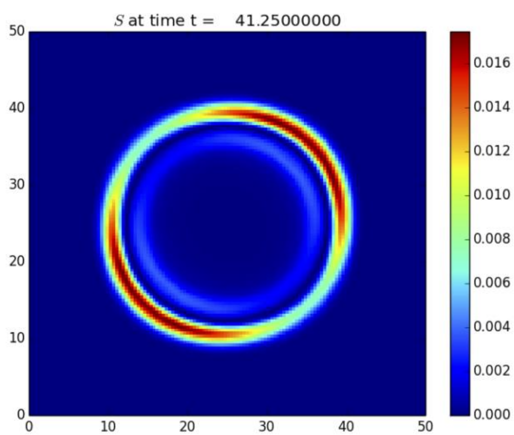


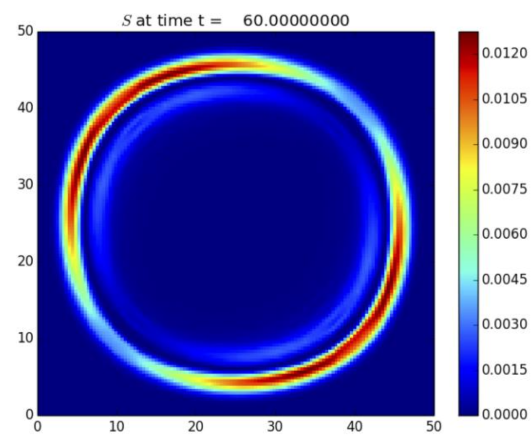
(a) $t = 3.75$



(b) $t = 18.75$



(c) $t = 41.25$



(d) $t = 60.0$

Summary

Innovation:

- Novel N-dimensional FVM scheme
- Redrawn link between wave speed and material-induced change of space geometry
- Generalized problem treatment
- Multidisciplinary solver with application to all wake types.

Impact:

- Fastest scheme and supports N-dimensions
- Second order accurate
- Highly scalable and 3D- ready
- Applications in photonics have been demonstrated

Questions

Thank you

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