Evolving Systems: An Introduction

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References


What It Is?

Evolving Systems=
Autonomously
Assembled
Active Structures

Or Self-Assembling
Structures,
which Aspire to a
Higher Purpose;
Cannot be attained
by Components Alone
How It Works

Evolved System

Controller 1

Controller 2

Susan’s Slide
The Process of Evolving Systems

Active Component 1

Active Component 2

Active Component 3

Mated Components

Evolved System
It’s not theories about stars; it’s the actual stars that count.”

......... Freeman Dyson
Evolving Systems Applications

- Autonomous Assembly in Space

International Space Station after 9 December 2006 Mission
Evolving Systems Applications

- Autonomous Rendezvous and Docking
- Servicing and System Upgrades

DARPA’s Orbital Express
(ASTRO Servicing Satellite pictured on left)
Stability is Essential During the Entire Evolution Process

Orion Crew Exploration Vehicle Docking with the ISS
Launch Vehicles: Devolving Systems

NASA-MSFC

Ares-Orion

$\$$
Constellations and Formations of Spacecraft (NASA-JPL)

Separated Spacecraft Interferometers
The next step in network-centric warfare will be the creation of networked sensing suites that tailor their observations to the adversary’s rate of activity. These various sensors will concentrate on observing changes rather than on observing scenery …

Signal Magazine
Evolving Spacecraft Formations

Joining Spacecraft
“It is surprising how quickly we replace a human operator with an algorithm and call it SMART”
Affine Nonlinear Systems

\[
\begin{cases}
\dot{x} = A(x) + B(x)u \\
y = C(x)
\end{cases}
\]

With smooth vector fields: \((A(x), B(x), C(x))\)
defined on a neighborhood a smooth manifold, or \(\mathbb{R}^N\)
Evolving Systems: General

\[ \begin{aligned}
&\text{i th Component} \\
&\dot{x}_i = A_i(x_i) + B_i(x_i)u^c_i \\
y_i = C_i(x_i)
\end{aligned} \]

\[ \Rightarrow \begin{aligned}
&\dot{x} = A(x) + B(x)u \\
y = C(x)
\end{aligned} \]

where \[ \dot{x}_i = A_i(x_i) + B_i(x_i)u_i + \sum_{j=1}^{L} \epsilon_{ij} A_{ij}(x_i, x_j, u_j); \]

Local Controller:
\[ \begin{aligned}
u_i^c &= h_i(z_i) + u_i \\
\dot{z}_i &= l_i(z_i, y_i, u_i)
\end{aligned} \]

Evolved System with \[ x = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}, \]

\[ 0 \leq \epsilon_{ij} \leq 1 \]

\[ \epsilon_{ij} = \epsilon_{ji} \]
Evolving Systems: 2 Components

\[ 0 \leq \varepsilon_{12} = \varepsilon_{21} \equiv \varepsilon \leq 1 \]

Component 1

\[
\begin{align*}
\dot{x}_1 &= A_1(x_1) + B_1(x_1)u_1 + \varepsilon A_{12}(x_1, x_2, u_2) \\
y_1 &= C_1(x_1)
\end{align*}
\]

Component 2

\[
\begin{align*}
\dot{x}_2 &= A_2(x_2) + B_2(x_2)u_2 + \varepsilon A_{21}(x_2, x_1, u_1) \\
y_2 &= C_2(x_2)
\end{align*}
\]

\[ \Rightarrow \text{Evolved System} \]

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} A(x) \\ A_1(x_1) \\ A_2(x_2) \end{bmatrix} + \begin{bmatrix} B_1(x_1) & 0 \\ 0 & B_2(x_2) \end{bmatrix} u + \varepsilon \begin{bmatrix} A_{12}(x_1, x_2, u_2) \\ A_{21}(x_2, x_1, u_1) \end{bmatrix} \\
y &= \begin{bmatrix} C_1(x_1) \\ C_2(x_2) \end{bmatrix} \equiv C(x)
\end{align*}
\]

\[ \varepsilon = 0 \text{ unconnecte d} \]

\[ \varepsilon = 1 \text{ fully connected} \]
Evolving Systems: Evolution=Homotopy

Evolved System

\[ 0 \leq \varepsilon \leq 1 \]
Genetics of Evolving Systems: Inheritance of Component Traits

- Controllability/Observability
- **Stability**
- **Dissipativity**
- Optimality
- Robustness
- Disturbance Rejection/Signal Tracking

Source: CNN.com
How To Analyze Evolving Systems

- Admittance/Impedance
- Perturbation Methods
- Graph Theory
- Differential Geometry - Lie Groups
- Other stuff
Impedance and Admittance of Components

\[ v_1 \equiv \dot{x}_1 = \dot{x}_2 \]
\[ x_1 = x_2 \]

Component 1 \hspace{1cm} Component 2

\[ f_2 = -f_1 \]

\[ Z \equiv \text{Impedance Operator} : \quad F = Z(V) \]
\[ Y \equiv \text{Admittance Operator} : \quad V = Y(F) \]

\[ Z \equiv Y^{-1}, \quad Y \equiv Z^{-1} \]

Fully Evolved System

\[ f_1 = -f_2 \]

\[ v_1 = Y_1(f_1) \]

\[ v_1 = \dot{x}_1 \]

\[ f_2 = Z_2(v_2) \]

\[ \dot{x}_2 = \dot{x}_1 \]
Nonlinear State Space Version: Admittance-Impedance

\[
\begin{align*}
\dot{x}_1 &= A_1(x_1) + \varepsilon B_1(x_1)u_1 + B_1^A(x_1)u_1^A \\
y_1 &= C_1(x_1) \\
y_1^A &= C_1^A(x_1) \\
\end{align*}
\]

Component 1

\[
\begin{align*}
\dot{x}_2 &= A_2(x_2) + \varepsilon B_2(x_2)u_2 + B_2^A(x_2)u_2^A \\
y_2 &= C_2(x_2) \\
y_2^A &= C_2^A(x_2) \\
\end{align*}
\]

Component 2

\[
\begin{align*}
u_1 &= -u_2 \\
u_2 &= y_1 \\
\end{align*}
\]
Characterize **LINEAR** Evolved System Stability

Q: Given System 1, can we find all System 2 so that connected system is stable?

Use Youla Parametrization for *nonproper* Impedances/Admittances.
Youla or Q Parameterization

Component 1

\[ Y_1u = ND^{-1}u \]
where \( N, D \) are coprime and stable

Component 2

\[ Z_2 = (Y - QN)^{-1}(X + QD) \]
Let \( X, Y \) be coprime and stable, and \( NX + DY = I \), and \( Q \) be any stable rational function.

All controllers that make the closed-loop stable.

M. Vidyasagar,
Control System Synthesis: A Factorization Approach,
MIT Press, 1985
Dissipativity: 
“Higher” Form of Stability

Energy Stored $\leq$ Energy Supplied
 e.g. Springs and Masses
 Inductors and Capacitors
 Roommates/Spouses
Definition of Dissipativity

Energy Storage Function:
\[
V(x) > 0; \forall x \neq 0; \\
V(0) = 0
\]

System is Dissipative when
\[
\begin{align*}
L_{A(x)} V & \equiv \nabla V A(x) \leq 0 \\
L_{B(x)} V & \equiv \nabla V B(x) = C^T (x)
\end{align*}
\]

⇒ \( \frac{dV(x(t))}{dt} \equiv \nabla V \dot{x}(t) = \nabla V [A(x) + B(x)u] \)

\( \leq 0 + C^T(x)u = y^T(t)u(t) \equiv \langle y, u \rangle \)

or

\[
V(x(t)) \leq V(x(0)) + \int_0^t \langle y, u \rangle d\tau
\]
Strict Dissipativity

Energy Storage Function : \[
\begin{align*}
V(x) &> 0; \forall x \neq 0; \\
V(0) &= 0
\end{align*}
\]

System is Strictly Dissipative when

\[
\begin{align*}
L_A(x)V &\equiv \nabla V A(x) \leq -S(x) \\
L_B(x)V &\equiv \nabla V B(x) = C^T(x)
\end{align*}
\]

\[
\Rightarrow \frac{dV(x(t))}{dt} = \nabla V \dot{x}(t) = \nabla V [A(x) + B(x)u]
\]

Internal Power Dissipated

\[
\leq -S(x) + C^T(x)u = \langle y, u \rangle - S(x)
\]
**Positive Realness (PR)**

Given \((A, B, C)\) and \(T(s) = C(sI - A)^{-1}B + D\):

There is \(P > 0\) so that

\[
\begin{align*}
K - Y \left\{ \begin{array}{l}
A^TP + PA = -L^TL \equiv Q \leq 0 \\
PB = C^T
\end{array} \right.
\]

\[\text{Re} T(j\omega) \equiv (T(j\omega) + T^*(j\omega))/2 \geq 0 \text{ for all } \omega\]

This is often taken as the definition of **Positive Real (PR)**

For Linear Systems: PR = Dissipative = Passive

**Linear & Dissipative**

**A Frequency Domain Condition**
What does **Strictly** Positive Real (SPR) mean?

Answer: Lotsa Things, Not all equivalent!! (See J.Wen, AC-33, 1988)

Here is ONE Definition of SPR:

There is $\mu > 0$ so that

$$\text{Re} T(j\omega - \mu) \equiv (T(j\omega - \mu) + T^*(j\omega - \mu))/2 \text{ is PR}$$

Relation to K - Y:

Just Change $A$ to $A + \mu I$ in K - Y

$$\begin{cases} A^T P + PA = -L^T L - 2\mu P \equiv -Q < 0 \\ PB = C^T \end{cases}$$

For Linear Systems: SPR=Strictly Dissipative=Strictly Passive
Almost Strictly Dissipative (ASD)

Nonlinear System

\[
\begin{align*}
\dot{x} &= A(x) + B(x)u \\
y &= C(x)
\end{align*}
\]

\[
\therefore (A_C(x) \equiv A(x) + B(x)GC(x), B(x), C(x)) \text{ is ASD}
\]

For LTI Systems: ASD=ASPR
Linear ASPR via Non-Orthogonal Projections

Balas & Fuentes:

1) (A,B,C) Almost SPR if and only if CB positive definite and open-loop transfer function $P(s) \equiv C(sI - A)^{-1}B$ is minimum phase (i.e. all transmission zeros stable).

2) Almost PR if and only if CB positive definite, open-loop transfer function is weakly minimum phase (i.e. can have some marginally stable transmission zeros), and marginally stable zero dynamics are PR.
Inheritance of Almost Strictly Dissipative Property

\[ \begin{align*}
    \dot{x}_1 &= A_1(x_1) + \varepsilon B_1(x_1)u_1 + B_1^A(x_1)u_1^A \\
y_1 &= C_1(x_1) \\
y_1^A &= C_1^A(x_1)
\end{align*} \]

Component 1

\[ \begin{align*}
    \dot{x}_2 &= A_2(x_2) + \varepsilon B_2(x_2)u_2 + B_2^A(x_2)u_2^A \\
y_2 &= C_2(x_2) \\
y_2^A &= C_2^A(x_2)
\end{align*} \]

Component 2

\[ u_1 = -u_2 \]

\[ u_2 = y_1 \]
Two Component Flexible Structure Evolving System

Example where stability is lost during evolution will follow!
System That Does Not Inherit Stability

**Component 1:**
\[
\begin{align*}
m_1 \ddot{q}_1 &= u_1 - \varepsilon k_{12} (q_1 - q_2) \\
y_1 &= [q_1, \dot{q}_1]^T
\end{align*}
\]

**Component 2:**
\[
\begin{align*}
m_2 \ddot{q}_2 &= u_2 - k_{23} (q_2 - q_3) - \varepsilon k_{12} (q_2 - q_1) \\
m_3 \ddot{q}_3 &= u_3 - k_{23} (q_3 - q_2) \\
y_2 &= [q_2, \dot{q}_2]^T \\
y_3 &= [q_3, \dot{q}_3]^T
\end{align*}
\]

Component Controllers:
\[
\begin{align*}
u_1 &= -(0.9s + 0.1)q_1 \\
u_2 &= -\left(\frac{0.1}{s} + 0.2s + 0.5\right)q_2 \\
u_3 &= -(0.6s + 1)q_3
\end{align*}
\]

with \(0 \leq \varepsilon \leq 1, m_1 = 30, m_2 = 1, m_3 = 1, k_{12} = 4\) and \(k_{23} = 1\)
Partial or Failed Evolutions:
Evolving Systems: Completely Connected
Evolving Systems: Failure to Connect

Subsystem 3 should connect to System 2, but did not!!!!!!
Key Component Evolving Controller

Local Controllers 1 … N and their components’ input-output ports remain unchanged
Closed Loop Poles of Example

No Evolving Controllers

System with Key Component Evolving Controller
Benefits of Key Component Evolving Controllers

- Maintains stability during Evolution
- Interchangeability of non-key components
- Cost savings & risk mitigation
- No clear design methodology
ADAPTIVE Key Component Evolving Controller

ADAPTIVE Key Component Controller

Component 1
Key Component

Component 2

... Component N

Controller 1

Controller 2

Controller N
Direct Adaptive Model Following Control (Wen-Balas 1989)

\[ u_m \rightarrow G_m \rightarrow y_m \]

\[ G_u \rightarrow u \rightarrow G_e \rightarrow y \]

Adaptive Gain Laws

\((u_m, x_m, e_y)\) Known Signals

\[ t \rightarrow \infty \rightarrow 0 \]
Direct Adaptive Persistent Disturbance Rejection (Fuentes-Balas 2000)

\[ u_m \rightarrow \text{Reference model} \rightarrow y_m \]

\[ \text{Disturbance Generator} \rightarrow \text{Plant} \rightarrow x_m \]

\[ u \rightarrow \text{Plant} \rightarrow y \]

\[ (u_m, x_m, e_y, \phi_2) \text{ Known Signals} \]

\[ G_e \]

\[ e_y \rightarrow 0 \text{ as } t \rightarrow \infty \]
Adaptive Key Component Control

\[ u_1 = -u_2 \]

\[ \dot{x}_1 = A_1(x_1) + \varepsilon B_1(x_1)u_1 + B_1^A(x_1)u_1^A \]
\[ y_1 = C_1(x_1) \]
\[ y_1^A = C_1^A(x_1) \]

Key Component 1

Strictly Dissipative

\[ \dot{x}_2 = A_2(x_2) + \varepsilon B_2(x_2)u_2 \]
\[ y_2 = C_2(x_2) \]

All Other Components
A Theorem Is Worth A Thousand Simulations
(Despite its limitations)
Adaptive Key Component Controller Theorem

If Key Component 1 \((u_1^A, y_1^A)\) is Almost Strictly Dissipative and Component 2 \((u_2, y_2)\) is Strictly Dissipative, then

\[
\begin{align*}
\text{Adaptive Key Component Controller} & \quad \begin{cases} 
  u_1^A = G_1 y_1^A \\
  \dot{G}_1 = - y_1^A (y_1^A)^T \gamma_1; \gamma_1 > 0
\end{cases} \\
\text{Produces} & \quad x_1, x_2 \xrightarrow{t \to \infty} 0 \text{ and } G_1 \text{ is bounded} \\
\text{throughout the entire evolution} & \quad 0 \leq \varepsilon \leq 1
\end{align*}
\]
Now Let’s See Some Detailed Mathematical Proofs

No No Please, I’d Rather Be Eaten Alive by Radioactive Spiders
Let \( V_2 \equiv \frac{1}{2} \text{trace} \Delta G \cdot \Delta G^T > 0 \)

\[ \Rightarrow \dot{V}_2 = \text{trace} \Delta \dot{G}_1 \gamma_1^{-1} \Delta G_1^T = \text{trace}(y_1^A (y_1^A)^T \gamma_1 \gamma_1^{-1} \Delta G^T) \]

\[ = (\Delta G y_1^A)^T y_1^A = \langle y_2, u_2 \rangle \quad \therefore \text{System 2 is dissipative} \]
Adaptive Key Component Control Can Mitigate Persistent Disturbances Also

\[
\begin{align*}
\dot{x}_1 &= A_1(x_1) + \varepsilon B_1(x_1)u_1 + B_1^A(x_1)u_1^A + \sum_{i=1}^{n} u_{D_i} \\
y_1 &= C_1(x_1) \\
y_1^A &= C_1^A(x_1)
\end{align*}
\]

Key Component 1

\[
\begin{align*}
\dot{x}_2 &= A_2(x_2) + \varepsilon B_2(x_2)u_2 \\
y_2 &= C_2(x_2)
\end{align*}
\]

All Other Components

\[
\begin{align*}
u_1 &= -u_2 \\
\end{align*}
\]
Adaptive Key Component Control of Nonlinear Plant

Mass Displacement with No Adaptive Control

Mass Displacement with Adaptive Control on Mass 1
Adaptive Key Component Controller

Component 1 is Key Component w/ I/O ports on mass 1

Component 2 is Key Component w/ I/O ports on mass 2

Nonminimum phase zeros:
0.0051466 + 0.20089i, 
0.0051466 - 0.20089i
Evolving Spacecraft Formations

Joining Spacecraft

Are You Lookin’ at Me?
Control Based on Relative State

\[ u_2 = G \xi_1 \]

\[ \xi_1 \equiv x_2 - x_1 - r_1 \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \]

\[ u_2 = G \xi_1 \]

\[ = \begin{bmatrix} g_P & g_D \end{bmatrix} \begin{bmatrix} q_2 - q_1 - r^1_P \\ \dot{q}_2 - \dot{q}_1 - r^1_D \end{bmatrix} \]

\[ = g_P (q_2 - q_1 - r^1_P) + g_D (\dot{q}_2 - \dot{q}_1 - r^1_D) \]

Uni-directional Newton’s 3rd Law
Reciprocity = Usual Newton’s Laws

\[ u_2 = G \xi \]

\[ u_1 = -G \xi = -u_2 \]

\[ \xi_1 \equiv x_2 - x_1 - r_1 \xrightarrow{t \to \infty} 0 \]

\[ u_2 = G \xi_1 = -u_1 \]

\[ = g_p (q_2 - q_1 - r^1_p) + g_D (\dot{q}_2 - \dot{q}_1 - r^1_D) \]

- preloaded spring
- preloaded damper
Spacecraft Dynamics

\[ SC_k : \dot{x}_k = Ax_k + Bu_k + \Gamma u^D_k; \quad k = 1,\ldots, N \]

Disturbance Generator:
\[
\begin{align*}
\dot{u}^D_k &= \theta \dot{z}_k^D \\
\dot{z}_k^D &= Fz_k^D \quad \text{or} \quad z_k^D = L_k \phi_D
\end{align*}
\]

Double Integrator
\[
A \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C \equiv [1 \quad 0.1]
\]
Open Chain Formation

SC1

SC2

SC3

SC4
Relative Measurements

$$SC_k : y_k = C \xi_{k-1} \equiv C(x_k - x_{k-1} - r_{k-1})$$

Chained Output Feedback:

$$\begin{cases} u_k = G y_k + G_D z_k^D = GC \xi_{k-1} + G_D z_k^D \\ u_1 = 0 \end{cases}$$

$$\Rightarrow \Delta u_k \equiv u_{k+1} - u_k = G(y_{k+1} - y_k) = GC(\xi_k - \xi_{k-1}) + G_D \Delta z_k^D$$

$$\therefore \dot{\xi}_k = A \xi_k + B \Delta u_k + \Gamma \Delta u_k^D$$

$$= (A + BGC)\xi_k - BGC \xi_{k-1} + (BG_D + \Gamma \theta) \Delta z_k^D; k \geq 2$$

& $$\xi_1 = (A + BGC)\xi_1$$
Open Chain Stability

\[
\Rightarrow \dot{\xi}_k = (A + BGC)\xi_k - BGC\xi_{k-1}
\]

Let \( \xi \equiv \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \Rightarrow \dot{\xi} = \begin{bmatrix} A + BGC & 0 & 0 \\ -BGC & A + BGC & 0 \\ 0 & -BGC & A + BGC \end{bmatrix} \xi \\

\overline{A_c} \text{ stable } \Leftrightarrow A + BGC \text{ stable}
Open Chain Formation with Reciprocity

SC1 ➔ SC2

SC4 ↔ SC3
Stability Open Chain with Reciprocity

Output Feedback:

\[
\begin{align*}
    u_1 &= 0 \\
    u_2 &= GC \xi_1 + G_D z_2^D \\
    u_3 &= GC \xi_2 + G_D z_3^D + (-GC \xi_3) \\
    u_4 &= GC \xi_3 + G_D z_4^D
\end{align*}
\]

Let \( \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \) \( \Rightarrow \dot{\xi} = \begin{bmatrix} A + BGC & 0 & 0 \\ -BGC & A + BGC & -BGC \\ 0 & -BGC & A + 2BGC \end{bmatrix} \xi \)

\( \bar{A}_C \) stable \( \iff \) \( A + BGC \) and ?? \( \cdots \) \( \text{Can it be Unstable?} \)
**Theorem: Open Chain with Reciprocity**

\[ \overline{A_c} = \begin{bmatrix} A + BGC & 0 & 0 \\ -BGC & A + BGC & -BGC \\ 0 & -BGC & A + 2BGC \end{bmatrix} \]

\[ \overline{A_c} \]

\( \Leftrightarrow A + BGC \) and \( A + (2 + \alpha)BGC, A + (1 - \alpha)BGC \) stable

where \( \alpha = -\frac{1}{2} \pm \frac{1}{2} \sqrt{5} = 0.62, -1.62 \) (or \( \alpha^2 + \alpha - 1 = 0 \))

**Proof:**

\[
\begin{bmatrix} I & -\alpha I \\ I & 0 \end{bmatrix} \begin{bmatrix} I & \alpha I \end{bmatrix} = \begin{bmatrix} A + (2 + \alpha)BGC & (\alpha^2 + \alpha - 1)BGC \end{bmatrix}
\]

**Cor.** \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \ 0.1] \Rightarrow \overline{A_c} \) stable \( \forall G = g > 0 \)
Open Chain Formation

SC1

SC2

SC3

SC4
Theorem:

\[
\bar{A}_c = \begin{bmatrix}
A + BGC & 0 & 0 \\
-BGC & A + BGC & 0 \\
BGC & 0 & A + 2BGC
\end{bmatrix}
\]

\[\Rightarrow A + BGC \text{ and } A + 2BGC \text{ stable}\]

Cor. \(A \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B \equiv \begin{bmatrix} 0 \end{bmatrix}, C \equiv \begin{bmatrix} 1 & 0.1 \end{bmatrix} \Rightarrow \bar{A}_c \text{ stable } \forall G = -g < 0\]
Closed Chain Formation

SC1

SC2

SC3

SC4
\[ \bar{A}_C(\varepsilon) = \begin{bmatrix} A + (1+\varepsilon)BGC & \varepsilon BGC & \varepsilon BGC \\ -BGC & A + BGC & 0 \\ 0 & -BGC & A + BGC \end{bmatrix} \]

\[ = \begin{bmatrix} A + BGC & 0 & 0 \\ -BGC & A + BGC & 0 \\ 0 & -BGC & A + BGC \end{bmatrix} + \varepsilon \begin{bmatrix} BGC & BGC & BGC \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ \bar{A}_0 \text{ stable} \Leftrightarrow A + BGC \text{ and } A + 2BGC \text{ stable} \]

\[ \Rightarrow \exists \varepsilon_0 > 0 \exists, \forall 0 \leq \varepsilon < \varepsilon_0, \bar{A}_C(\varepsilon) \text{ is stable.} \]

**Cor.**
\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \ 0.1] \]

\[ \Rightarrow \bar{A}_C(\varepsilon) \text{ stable} \forall 0 \leq \varepsilon < \varepsilon_0 = 0.02(\text{theory} \sim 10^{-13}), \text{ and unstable for } \varepsilon = 1 \]
ADAPTIVE Key Component Controller

Was Unstable, but
Adaptive Key Component Controller Restores Stability
Example

Double Integrator: \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \), \( B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( C = \begin{bmatrix} 1 & 0.1 \end{bmatrix} \)

\[ C(s) \equiv -(10 + \frac{1}{s}) \]

\[ r_1 = \frac{3}{0}; r_2 = \frac{5}{0}; r_3 = \frac{8}{0}; r_4 = \frac{16}{0} \]

Key Component Adaptive Controller

\[ u_4 = GC\frac{\xi_3}{s} + G_e\ddot{y}_4 + G_D\phi_D; \phi_D \equiv 1 \]

\( \text{Original \ SC} \ 4 \ Control \quad \text{Adaptive \ Control} \)

Adaptive Gains

\[ \begin{align*}
\dot{G}_e &= -(\ddot{y}_4)^2 \gamma_e; \gamma_e \equiv 100 \\
\dot{G}_D &= -\ddot{y}_4 \phi_D \gamma_e; \gamma_e \equiv 1
\end{align*} \]

Measured Output: \( \ddot{y}_4 = Cx_4 \)
Conjecto-Theorem

If individual spacecraft dynamics \((A, B, C)\) are ASPR, ie \(CB > 0\) and \(P(s) = C(sI - A)^{-1}B\) is minimum phase, then

Key Component Adaptive Controller (on the joining spacecraft)

\[
u_N = GC\xi_{N-1} + G_e\tilde{y}_N + G_D\phi_D; \phi_D \text{ bounded}
\]

Original \(SC\ N\ Control\)

Adaptive Control

Adaptive Gains

\[
\begin{align*}
\dot{G}_e &= - (\tilde{y}_N)^2 \gamma_e; \gamma_e \equiv 100 \\
\dot{G}_D &= - \tilde{y}_N \phi_D \gamma_e; \gamma_e \equiv 1
\end{align*}
\]

Measured Output: \(\tilde{y}_N = Cx_N\)

produces \(\xi_k = x_{k+1} - x_k - r_k \xrightarrow{t \to \infty} 0 \forall k\)

(and rejects bounded disturbances)
with bounded adaptive gains
Proof: WLOG use 4 SC

Let \( \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \) ⇒
\[
\begin{align*}
\dot{\xi} &= A_C(\varepsilon)\xi + B u_4 \\
\tilde{y}_4 &= C_4 x_4 = \tilde{C} \xi
\end{align*}
\]

where \( A_C(\varepsilon) = \begin{bmatrix} A + (1 + \varepsilon)BG & \varepsilon BG & \varepsilon BG \\ -BG & A + BG & 0 \\ 0 & -BG & A + BG \end{bmatrix} \)

\[
\begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix} + \begin{bmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{bmatrix} \begin{bmatrix} (1 + \varepsilon)G & \varepsilon G & \varepsilon G \\ -G & G & 0 \\ -G & G & 0 \end{bmatrix} \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix}
\]
Find $G_{s} \in u_{4} = G_{s} \tilde{y}_{4}$

\[ \Rightarrow \tilde{A}_{C}(\varepsilon) = \tilde{A}_{C}(\varepsilon) + \begin{bmatrix} 0 \\ 0 \\ G_{s}[0 \ 0 \ C] \\ B \end{bmatrix} \]

\[ = \begin{bmatrix} A \ 0 \ 0 \\ 0 \ A \ 0 \\ 0 \ 0 \ A \end{bmatrix}_{\tilde{A}} + \begin{bmatrix} B \ 0 \ 0 \\ 0 \ B \ 0 \\ 0 \ 0 \ B \end{bmatrix}_{\tilde{B}} + \begin{bmatrix} (1 + \varepsilon)G \ \varepsilon G \ \varepsilon G \\ 0 \ -G \ G \\ 0 \ -G \ G \end{bmatrix}_{\tilde{G}(\varepsilon)} \begin{bmatrix} C \ 0 \ 0 \\ 0 \ C \ 0 \\ 0 \ 0 \ C \end{bmatrix}_{\tilde{C}} \]

\[ = \tilde{A} + \tilde{B}(\tilde{G} + G_{s})\tilde{C} \]

Clearly $\overline{C B} > 0 \Leftrightarrow CB > 0$

and $\overline{P}(s)$ minimum phase $\Leftrightarrow P(s)$ minimum phase.

\[ \therefore (A, B, C)ASPR \Rightarrow (\overline{A}, \overline{B}, \overline{C})ASPR \Rightarrow (\tilde{A}_{C}(\varepsilon), \tilde{B}, \tilde{C})ASPR# \]
Future Formation Stuff

• Nonlinear: \[
\begin{aligned}
\dot{x}_k &= Ax_k + g(x_k) + Bu_k \\
y_k &= Cx_k
\end{aligned}
\]
  Lipschitz & weak enough

• Harder but Doable

\[
\begin{aligned}
\dot{x}_k &= A(x_k) + B(x_k)u_k \\
y_k &= C(x_k)
\end{aligned}
\]

• Maintaining a Formation Shape (Tracking); Immutability of Formation Shapes

• Swarms: George Hill’s Eqs
  (or Johnny-Come-Lately: Clohessy-Wiltshire)
A Sortof Paradigm

Wonderful New Survivable System

Old Decrepit Broken-down System (eg US power grids)

New Exciting Adaptive Controller